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QUARTERLY OF APPLIED MATHEMATICS

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THE TORSION OF A HOLLOW SQUARE*

BY

J. L. SYNGE† AND W. F. CAHILL

National Bureau of Standards

Summary. SEAC‡ is used to give an approximate solution to the torsion problem for a hollow square, with accurate determination of lower and upper bounds for torsional rigidity.

1. The torsion problem for a multiply connected cross section. To solve the torsion problem for a beam of multiply connected cross section, we seek a function $\psi(x, y)$ which is harmonic in the domain A occupied by the material, and such that

$$\psi = \tfrac{1}{2}r^2 \text{ on the outer boundary } B_0 , \tag{1.1}$$

$$\psi = \tfrac{1}{2}r^2 + C_\lambda \text{ on the inner boundaries } B_\lambda ,$$

the constants C_λ being chosen so that for each inner boundary we have

$$\int_{B_\lambda} \frac{\partial \psi}{\partial n}\, dB_\lambda = 0. \tag{1.2}$$

Here $r^2 = x^2 + y^2$, where x, y are rectangular Cartesian coordinates in the plane of the cross section.

The components of shearing stress are

$$\mu\alpha \frac{\partial}{\partial y}(\psi - \tfrac{1}{2}r^2), \qquad -\mu\alpha \frac{\partial}{\partial x}(\psi - \tfrac{1}{2}r^2), \tag{1.3}$$

where μ is the rigidity and α the twist per unit length. The lines of stress are given by the equation

$$\psi - \tfrac{1}{2}r^2 = \text{const.}, \tag{1.4}$$

this family of curves including, of course, the curves B_0 , B_λ .

The warping of the section is given by $\phi(x, y)$, which is the harmonic function conjugate to ψ.

If we denote the torque by $\mu\alpha\Gamma$, then

$$\Gamma = I - \int \left[\left(\frac{\partial \psi}{\partial x} \right)^2 + \left(\frac{\partial \psi}{\partial y} \right)^2 \right] dA, \tag{1.5}$$

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†On leave of absence from the Dublin Institute for Advanced Studies.

‡Standards Eastern Automatic Computer.

when I is the polar moment of inertia of the cross section,

$$I = \int (x^2 + y^2)\, dA. \tag{1.6}$$

2. The torsion of a hollow square. The method used below for solving approximately the torsion problem for a hollow square cross section is essentially the hypercircle method, originally developed by Prager and Synge[1] for more general problems in elasticity.

Details of the method will be found in a forthcoming book,[2] together with numerical results obtained by a desk calculator. The present paper is concerned with the application of SEAC to these calculations, and only essential formulas will be given to make the computations intelligible.

The method is applicable in principle to any cross section, and is particularly suited to a cross section which is a hollow rectangle or a hollow square. The only case treated here in detail is the case of a hollow square bounded externally by a square B_0 of side s and internally by a square B_1 of side $\frac{1}{2}s$, as shown in Fig. 1.

The square is triangulated as shown, a good approximation demanding a fine triangulation. We speak of approximation of order n when the outer side is divided into n equal parts. On account of symmetry, only an octant is needed in the calculations; Fig. 1 shows the triangulated octant MNZY for $n = 16$.

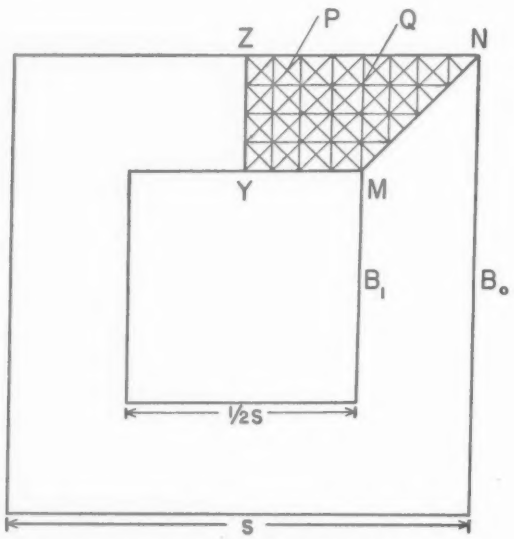

FIG. 1. Triangulation for the approximation $n = 16$.

Each junction in the triangulation is a "station." There are two types of station, type P at the centers of the small squares and type Q at the corners of these small squares.

[1] W. Prager and J. L. Synge, Quart. Appl. Math. 5 (1947), 241-269.

[2] J. L. Synge, *The hypercircle in mathematical physics*, Cambridge University Press.

To each station we attach a "weight" w. These weights uniquely determine a function $f(x, y)$ in the domain A of the cross section, by the rule that $f(x, y) = w$ at each station and $f(x, y)$ is a linear function of x and y in each triangle.

The general plan of approximation is to find two sets of weights, say w' and w'', such that if $f'(x, y)$ and $f''(x, y)$ are the corresponding functions, defined as above, then the following approximations hold:

$$\begin{aligned} f'(x, y) &\sim \psi - \tfrac{1}{2}r^2, \\ f''(x, y) &\sim \phi. \end{aligned} \tag{2.1}$$

This plan is slightly modified to permit dimensionless calculations, "reduced weights" b', b'' being introduced.

3. Approximation for lines of stress, and lower bound for torsional rigidity. To approximate $\psi - \frac{1}{2}r^2$ we have to solve the following difference equations:

$$\begin{aligned} b'(P) &= \tfrac{1}{4} \sum b'(Q) - d(P)\, b'(R) + 1, \\ b'(Q) &= \tfrac{1}{4} \sum b'(P) - d(Q)\, b'(R) + 2, \\ b'(R) &= -\frac{4}{3n^2} \sum_{MN} [b'(P) + b'(Q)] + \frac{7n}{12}, \\ b'(Q) &= 0 \text{ for } Q \text{ on } B_0 \text{ or } B_1 . \end{aligned} \tag{3.1}$$

Here $b'(P)$, $b'(Q)$ are reduced weights at stations P, Q. The interpretation of the first line of (3.1) is as follows: The $\sum$ is a sum over the four stations Q which are neighbors of P. $d(P) = 1$ if P is on MN, and otherwise $d(P) = 0$. $b'(R)$ is an additional reduced weight, introduced to allow for multiple-connectivity. The interpretation of the second line of (3.1) is similar. Note that if Q is on B_0 or B_1 it lacks some of its four neighbors; in that case fictitious neighbors are supplied with zero weights. In the third line of (3.1) $\sum_{MN}$ is summation along the diagonal MN of Fig. 1.

The weights $b'(P)$, $b'(Q)$ are symmetric with respect to the lines of symmetry of the hollow square.

The approximation to $\psi - \frac{1}{2}r^2$ is given by

$$\psi - \tfrac{1}{2}r^2 \sim \frac{1}{6}\frac{s^2}{n^2} F'(x, y) + \frac{1}{3}\frac{s^2}{n} b'(R) G(x, y), \tag{3.2}$$

where $F'(x, y)$ is a function which is linear in each triangle and takes, at the stations P, Q, the values $b'(P)$, $b'(Q)$, while $G(x, y)$ is a function which is linear in each of the four triangles in which the outer square is divided by its diagonals, with $G = 1$ at the center of the squares, and $G = 0$ on B_0 .

If we denote the torque by $\mu\alpha\Gamma$, then a lower bound for torsional rigidity is given by

$$\frac{\Gamma}{s^4} \geq \frac{1}{9n^4} [\sum b'(P) + 2 \sum b'(Q)] + \frac{7}{36n} b'(R), \tag{3.3}$$

these summations extending over the whole domain A, and not merely the octant MNZY.

4. Approximation for warping function, and upper bound for torsional rigidity. To approximate ϕ we have to solve

$$b''(P) = \tfrac{1}{4} \sum b''(Q),$$
$$b''(Q) = \tfrac{1}{4} \sum b''(P) + m(Q). \tag{4.1}$$

Here each summation is over four neighbors. If neighbors are lacking, i.e. when Q is on B_0 or B_1, fictitious neighbors are to be added by reflection in B_0 or B_1, the images having the same weights as the actual neighbors. The number $m(Q)$ is given by

$$m(Q) = 0 \text{ if } Q \text{ is inside the domain } A,$$
$$m(Q) = 2x/s \text{ if } Q \text{ is on } B_0 \text{ (the outer boundary)}, \tag{4.2}$$
$$m(Q) = -2x/s \text{ if } Q \text{ is on } B_1 \text{ (the inner boundary)}.$$

Here x is the abscissa of Q, the origin of coordinates being at the center of the squares.

In this case the weights are skew-symmetric; therefore

$$b''(P) = 0, \; b''(Q) = 0, \tag{4.3}$$

if P or Q is on a line of symmetry of the section.

The approximation to the warping function ϕ is given by

$$\phi \sim \frac{s^2}{4n} F''(x, y), \tag{4.4}$$

where $F''(x, y)$ is linear in each triangle and has the values $b''(P)$, $b''(Q)$ at the stations P, Q.

An upper bound for the torsional rigidity is given by

$$\frac{\Gamma}{s^4} \leq \frac{1}{n^2} \left[\frac{5n^2}{32} - \sum_{ZN} m(Q) b''(Q) - \sum_{YM} m(Q) b''(Q) \right], \tag{4.5}$$

the summations being along ZN and YM in Fig. 1.

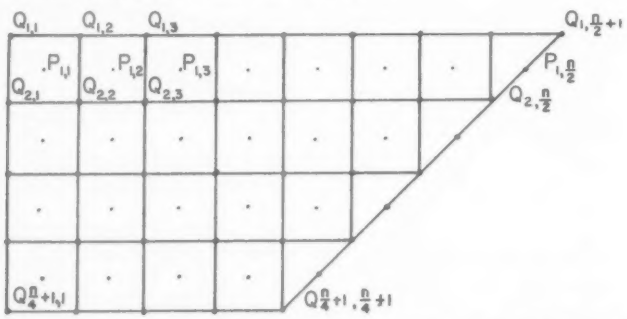

FIG. 2. Enumeration of stations for the approximation n (a multiple of 4)

5. Plan for SEAC calculations. For the approximation of order n, the stations were numbered as indicated in Fig. 2.[1] Then, using the letters P, Q, R for the reduced

[1]The procedure may be applied with little additional complication, to the case where the ratio of the sides s, s' of the outer and inner squares is not 2:1. If it is possible to triangulate with n squares along s and n' squares along s', then we make the following notational changes in Fig. 2:

$$Q_{(n/4)+1,1} \rightarrow Q_{(n-n')/2+1,1}$$
$$Q_{(n/4)+1,(n/4)+1} \rightarrow Q_{(n-n')/2+1,(n'/2)+1}.$$

weights b', we have, as in §3,

$$Q(x, y) = Q(y, x) = Q(-x, y) = Q(x, -y) = Q(-x, -y), \tag{5.1}$$

$$P(x, y) = P(y, x) = P(-x, y) = P(x, -y) = P(-x, -y), \tag{5.2}$$

$$R = -\frac{4}{3n^2} \sum_{MN} (P + Q) + \frac{7n}{12}, \tag{5.3}$$

$$P_{ij} = \tfrac{1}{4}(Q_{ij} + Q_{i,j+1} + Q_{i+1,j} + Q_{i+1,j+1}) + 1 - d(P_{ij})R, \tag{5.4}$$

$$Q_{ij} = \tfrac{1}{4}(P_{i-1,j-1} + P_{i-1,j} + P_{i,j-1} + P_{ij}) + 2 - d(Q_{ij})R, \tag{5.5}$$

$$d(X_{ij}) = \begin{cases} 1 \text{ if } X_{ij} \text{ sits on diagonal,} \\ 0 \text{ otherwise} \end{cases} \tag{5.6}$$

$$Q_{1j} = Q_{(n/4)+1,j} = 0. \tag{5.7}$$

The lower bound for Γ is given by

$$\left(\frac{\Gamma}{s^4}\right)_{LB} = \frac{1}{9n^4}\left\{\sum_{ALL} (P + 2Q) + \frac{7}{36n} R\right\}. \tag{5.8}$$

Using P, Q for the reduced weights b'', we have as in §4,

$$Q(x, y) = -Q(-x, y) = -Q(x, -y) = Q(-x, -y) = -Q(y, x), \tag{5.9}$$

$$P(x, y) = -P(-x, y) = -P(x, -y) = P(-x, -y) = -P(y, x). \tag{5.10}$$

For interior points,

$$P_{ij} = \tfrac{1}{4}(Q_{ij} + Q_{i,j+1} + Q_{i+1,j} + Q_{i+1,j+1}), \tag{5.11}$$

$$Q_{ij} = \tfrac{1}{4}(P_{i-1,j-1} + P_{i-1,j} + P_{i,j-1} + P_{i,j}); \tag{5.12}$$

on the diagonal

$$P_{ij} = Q_{ij} = 0; \tag{5.13}$$

on the outer boundary B_0,

$$Q_{1j} = \tfrac{1}{2}(P_{1,j-1} + P_{1,j}) + \frac{2(j-1)}{n}, \tag{5.14}$$

$$Q_{11} = Q_{1,(n/2)+1} = 0;$$

on the inner boundary B_1,

$$Q_{(n/4)+1,j} = \tfrac{1}{2}(P_{(n/4),j-1} + P_{(n/4),j}) - \frac{2(j-1)}{n}, \tag{5.15}$$

$$Q_{(n/4)+1,1} = Q_{(n/4)+1,(n/4)+1} = 0;$$

on the vertical axis,

$$Q_{i1} = 0. \tag{5.16}$$

The upper bound for Γ is given by

$$\left(\frac{\Gamma}{s^4}\right)_{UB} = \frac{5}{32} - \frac{1}{n^2}\left\{\sum_{k=1}^{(n/2)+1} Q_{1k}m(Q_{1k}) + \sum_{k=1}^{(n/4)+1} Q_{(n/4)+1,k}m(Q_{(n/4)+1,k})\right\}. \qquad (5.17)$$

The iteration scheme for the lower bound started with all $P_{ij} = Q_{ij} = 0$ and the following cycle describes one complete iteration:

1) Set $i = 1$
2) Compute all P in row i by formula (5.4)
3) Have we done last row of P?

No, go to step 4.
Yes, go to step 7.

4) Replace i by $i + 1$
5) Compute all Q in row i by formula (5.5)
6) Return to step 2
7) Compute R by formula (5.3)
8) Return to step 1

After every T iterations [where T (approximately 20) is chosen at the start of a problem] the approximations for some chosen P, for some chosen Q, for R and also the

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
32	32	32	32	31	31	30	29	27	26	24	21	19	15	12	8	3	-2	-8	-14	-21	-30	-40	-52	
63	63	62	62	61	60	58	56	54	51	47	43	38	32	26	18	9	0	-11	-23	-37	-53	-71	-92	
89	89	88	87	86	84	81	78	74	69	63	56	48	39	28	16	2	-13	-30	-49	-71	-96	-125		
114	113	112	112	110	108	105	101	96	91	84	75	66	53	40	25	8	-12	-33	-57	-84	-115	-150		
134	134	133	131	129	125	121	116	110	102	92	81	67	51	34	13	-9	-34	-63	-94	-130	-170			
153	152	152	150	148	145	140	135	128	120	109	97	82	64	44	22	-4	-33	-64	-100	-140	-184			
168	167	166	164	160	156	151	144	135	124	111	95	76	54	30	2	-30	-65	-104	-147	-196				
181	181	179	177	175	171	165	158	149	138	125	108	89	66	39	9	-25	-62	-104	-151	-202				
190	189	188	185	181	176	170	161	150	137	120	100	76	48	17	-19	-59	-103	-153	-207					
198	197	195	194	190	186	179	171	161	148	131	111	87	59	26	-11	-53	-100	-151	-208					
201	200	198	196	191	186	178	169	156	140	121	97	68	34	-4	-47	-96	-149	-208						
203	203	201	199	195	190	184	175	163	148	130	106	78	44	5	-40	-89	-144	-205						
201	200	198	195	191	185	177	167	153	136	114	86	53	13	-32	-83	-139	-201							
197	197	196	193	190	185	178	169	157	141	120	94	62	23	-23	-74	-132	-195							
190	189	187	184	180	175	167	156	143	124	100	69	31	-14	-66	-125	-189								
181	180	179	177	174	169	163	154	142	126	105	77	41	-4	-56	-115	-181								
168	167	166	163	159	154	148	137	124	106	81	48	6	-46	-106	-173									
153	152	151	150	147	143	137	130	119	105	84	55	16	-35	-95	-162									
134	133	132	130	127	123	118	110	99	83	59	25	-23	-83	-152										
113	113	112	111	109	106	102	97	89	77	60	33	-9	-68	-138										
89	89	88	87	85	82	78	73	66	54	36	4	-50	-123											
63	62	62	61	60	59	57	54	49	43	33	15	-27	-101											
32	32	32	31	31	30	29	27	24	20	13	-2	-63												
0	0	0	0	0	0	0	0	0	0	0	0	0												

TABLE I. Computation of lower bound in approximation $n = 48$. The numbers are solutions of Eq. (3.1), or equivalently (5.1) — (5.7), divided by 10 and rounded off.
$R = 30.13171$

sum of the absolute values of the residuals are printed. When this sum becomes sufficiently small, all values are printed.

The iteration scheme for the upper bound is similar to that of the lower bound.

6. **Numerical results.** The equations (3.1), or equivalently (5.1)-(5.7), were solved as indicated in §5, in the approximations $n = 8$, $n = 16$, $n = 32$, $n = 48$. This last was the highest approximation feasible on SEAC, without resorting to tape storage.

The solutions for $n = 48$ rounded off to a maximum of three significant figures, are exhibited in Table I. To within a constant factor, these numbers are the values of the function $F'(x, y)$ [cf. (3.2)] at the stations. Addition of the last term in (3.2) to the numbers in Table I yields the numbers in Table II, which therefore represent, to within a constant factor, the stress function $(\psi - \frac{1}{2}r^2)$. A plot of lines of stress could easily be prepared from Table II.

Similarly, the equations (4.1), or equivalently (5.9)-(5.16), were solved for the same approximations. The solutions for $n = 48$, rounded off to a maximum of three

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
92	92	92	92	91	91	90	89	87	86	84	81	79	75	72	68	63	58	52	46	39	30	20	8	
183	183	182	182	181	180	178	176	174	171	167	163	158	152	146	138	129	120	109	97	83	63	49	28	
270	270	269	268	267	265	262	259	255	250	244	237	229	220	209	197	183	168	151	133	110	85	56		
355	354	354	353	351	349	346	342	337	332	325	316	306	294	281	266	249	229	208	184	157	126	91		
435	435	434	432	430	426	422	417	411	403	393	382	368	352	335	314	292	267	238	207	171	131			
515	515	514	512	510	507	502	497	490	482	471	459	444	426	406	384	358	329	298	262	222	178			
590	589	588	586	582	578	573	566	557	546	533	517	498	476	452	424	392	357	318	275	226				
663	663	661	659	657	653	647	640	631	620	607	590	571	548	521	491	457	420	378	331	280				
732	731	730	727	723	718	712	703	692	679	662	643	618	590	559	523	483	439	389	335					
801	800	799	795	793	789	783	774	764	751	734	714	690	662	629	592	550	503	452	395					
869	863	861	859	854	849	841	832	819	803	784	760	731	697	658	616	567	514	455						
926	926	924	922	918	913	907	898	886	871	853	829	801	767	728	683	634	579	518						
984	983	981	978	974	968	960	950	936	919	897	869	836	796	751	700	644	582							
1041	1041	1040	1037	1034	1029	1022	1013	1001	985	964	938	906	867	821	770	712	649							
1094	1093	1091	1088	1084	1079	1071	1060	1047	1028	1004	973	935	890	838	779	715								
1145	1144	1143	1141	1138	1133	1127	1118	1106	1090	1069	1041	1005	960	908	849	783								
1192	1191	1189	1187	1183	1178	1171	1161	1148	1130	1105	1072	1030	973	918	851									
1238	1237	1236	1235	1232	1228	1222	1215	1204	1190	1169	1140	1101	1050	990	923									
1279	1278	1277	1275	1272	1268	1263	1255	1244	1228	1204	1170	1122	1062	993										
1318	1318	1317	1316	1314	1311	1307	1302	1294	1282	1265	1238	1196	1137	1067										
1355	1355	1354	1353	1351	1348	1344	1339	1332	1320	1302	1270	1216	1143											
1389	1388	1388	1387	1386	1385	1383	1380	1375	1369	1359	1341	1299	1225											
1418	1418	1418	1417	1417	1416	1415	1413	1410	1406	1399	1384	1318												
1450	1450	1450	1450	1450	1450	1450	1450	1450	1450	1450	1450	1450												

TABLE II. Computation of lower bound in approximation $n = 48$. The numbers represent approximate values of
$[(6 \times 48)^2/s^2]\ (\psi - \frac{1}{2}r^2)$

significant figures, are exhibited in Table III. These numbers represent, to within a constant factor, the values of the warping function ϕ, and from these numbers a plot of level lines of warping could easily be prepared.

0	63	126	188	249	309	367	422	475	523	567	606	638	662	677	683	677	658	626	577	510	423	313	174	0
	30	88	147	204	261	316	368	418	465	507	544	575	600	616	622	618	602	573	528	467	387	285	158	0
0	55	110	164	217	268	318	366	410	451	487	517	541	558	566	563	550	524	484	428	354	260	143	0	
	26	76	127	176	225	272	317	359	398	433	463	487	504	512	512	501	478	442	391	324	238	131	0	
0	47	94	140	185	229	272	312	349	382	412	435	453	463	464	455	436	403	357	296	217	119	0		
	22	65	103	140	181	230	263	303	335	364	387	405	416	419	412	396	367	326	270	198	109	0		
0	39	79	117	155	192	228	261	291	319	342	360	372	376	372	358	334	297	246	181	100	0			
	18	54	89	124	158	190	222	250	277	299	317	330	336	334	323	302	269	224	165	91	0			
0	32	64	96	126	157	185	212	237	259	277	290	298	298	290	272	244	204	150	83	0				
	14	43	71	99	126	152	177	200	221	239	253	262	264	258	244	220	184	136	75	0				
0	25	50	74	98	122	145	166	186	203	217	227	231	228	217	197	166	123	68	0					
	11	32	53	74	95	115	135	153	169	183	194	200	199	191	175	148	111	62	0					
0	18	36	53	71	88	105	122	137	151	162	169	171	167	154	132	99	56	0						
	7	22	39	50	65	79	93	107	120	131	139	144	142	133	116	88	50	0						
0	11	22	33	44	55	67	78	90	101	110	116	118	113	100	77	44	0							
	4	11	18	26	34	43	52	62	71	81	87	93	93	84	67	39	0							
0	4	8	12	16	22	28	35	43	52	61	68	72	68	56	33	0								
0	0	1	2	3	6	10	16	24	32	42	49	51	45	28	0									
0	−4	−7	−10	−12	−12	−12	−9	−4	3	13	24	32	32	22	0									
−4	−11	−18	−24	−28	−32	−33	−31	−26	−17	−4	10	18	19	0										
0	−11	−22	−32	−41	−48	−54	−56	−55	−48	−35	−17	0	8	0										
−7	−22	−36	−50	−62	−72	−80	−82	−80	−70	−51	−24	−3	0											
0	−19	−37	−55	−72	−86	−98	−107	−111	−107	−91	−59	−20	0											
−11	−34	−56	−78	−97	−115	−129	−138	−141	−134	−110	−71	0												
0	−27	−54	−79	−104	−127	−147	−163	−173	−175	−164	−127	0												

TABLE III. Computation of upper bound in approximation $n = 48$. The numbers are solutions of Eq.(4.1), or equivalently (5.9) — (5.16), multiplied by 100 and rounded off, they represent approximate values of $100 \times [(4 \times 48)/s^2]\,\phi$.

The bounds on torsional rigidity, and the number of iterations required to give solutions to nine significant figures (starting from zero values) in the several approximations, are shown in Table IV.

TABLE IV

Lower and upper bounds for Γ/s^4

n	Lower Bound	Upper Bound	(U.B.)—(L.B.)	Number of Iterations L.B.	U.B.
8	.1263 0626	.1320 1773	.0057 1147	50	60
16	.1282 4044	.1300 3396	.0017 9352	180	270
32	.1288 4044	.1294 2496	.0005 8452	680	1070
48	.1289 7704	.1292 8697	.0003 0993	1400	2200

In the approximation n, the number of simultaneous equations to be solved is $3n^2/16$ in the case of the lower bound, and one less in the case of the upper bound. Thus, for $n = 48$, there were 432 and 431 equations respectively.

REMOVAL OF ILL-CONDITIONING FOR MATRICES*

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Objective. The most common problem in numerical computation is the solution of simultaneous linear equations. The most frequent obstacle is ill-conditioning of the matrix of coefficients, resulting in substantial loss of accuracy. The present paper describes a method of overcoming this difficulty as soon as encountered—with a minimum of computational effort, preservation of reasonable accuracy, and emergence of a well-conditioned system.

The present method applies only to cases in which the equations describe a well-behaved system but are poorly specified in the sense that one or a few of the given equations are almost a linear combination of other equations. The method does *not* apply to cases like the Hilbert matrices, in which *every* row is nearly a linear combination of other rows.

To avoid encumbrance by irrelevant details and to fix ideas, we shall illustrate by a matrix A of order $n = 5$ throughout.

The mathematical key. With high-speed computers, the preferred method of solution is some variant of the Gauss elimination procedure. After k stages, A has been reduced to a form $R^{(k)}$ which, for $k = 2$, is illustrated by

$$R^{(2)} = \begin{bmatrix} 1 & r_{12} & r_{13} & r_{14} & r_{15} \\ & 1 & r_{23} & r_{24} & r_{25} \\ & & s_{33} & s_{34} & s_{35} \\ & & s_{13} & s_{44} & s_{45} \\ & & s_{53} & s_{54} & s_{55} \end{bmatrix}.$$

In general we can write

$$R^{(k)} = \begin{pmatrix} R_{11} & R_{12} \\ 0 & S \end{pmatrix}, \quad \text{where} \quad \begin{cases} R_{11} = \text{upper triangular } k \times k, \\ S = S^{(k)} = (n-k) \times (n-k). \end{cases}$$

Retracing backwards the steps of elimination, matrix $R^{(1)}$ is retrieved from $R^{(2)}$ as

$$R^{(1)} = T^{(2)}R^{(2)}, \quad \text{where} \quad T^{(2)} = \begin{bmatrix} 1 & & & & \\ 0 & l_{22} & & & \\ 0 & l_{32} & 1 & & \\ 0 & l_{42} & 0 & 1 & \\ 0 & l_{52} & 0 & 0 & 1 \end{bmatrix},$$

the elements l_{i2} of column 2 being precisely the elements $s_{i2}^{(1)}$ in the leftmost column of

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$S^{(1)}$. Continuing in this manner, matrix A is expressed in the form

$$A \equiv R^{(0)} = T^{(1)}T^{(2)} \cdots T^{(k)}R^{(k)} = L^{(k)}R^{(k)},$$

where $L^{(k)} = T^{(1)}T^{(2)} \cdots T^{(k)}$ is illustrated, for $k = 2$, by

$$L^{(2)} = T^{(1)}T^{(2)} = \begin{bmatrix} l_{11} & & & & \\ l_{21} & 1 & & & \\ l_{31} & 0 & 1 & & \\ l_{41} & 0 & 0 & 1 & \\ l_{51} & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & & \\ 0 & l_{22} & & & \\ 0 & l_{32} & 1 & & \\ 0 & l_{42} & 0 & 1 & \\ 0 & l_{52} & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} l_{11} & & & & \\ l_{21} & l_{22} & & & \\ l_{31} & l_{32} & 1 & & \\ l_{41} & l_{42} & 0 & 1 & \\ l_{51} & l_{52} & 0 & 0 & 1 \end{bmatrix}.$$

Similarly, in general,

$$L^{(k)} = T^{(1)}T^{(2)} \cdots T^{(k)} = \begin{pmatrix} L_{11} & 0 \\ L_{21} & I \end{pmatrix}, \quad \text{where} \quad \begin{matrix} L_{11} = \text{lower triangular } k \times k \\ I = (n-k) \times (n-k) \end{matrix}$$

As seen above, the elements l_{ik} are furnished directly by the elimination procedure: the leftmost column of $S^{(k)}$ always becomes, at the next stage, the new column $k+1$ of $L^{(k+1)}$.

It is noteworthy that matrices $L^{(k)}$, $R^{(k)}$ and the factorization $A = L^{(k)}R^{(k)}$ represent a generalization of the usual formula $A = LR \equiv L^{(n)}R^{(n)}$ in which L and R are postulated to be triangular.

We shall subsequently also need $P^{(k)}$ = inverse of $L^{(k)}$, which can be written

$$P^{(k)} = \begin{pmatrix} P_{11} & 0 \\ P_{21} & I \end{pmatrix}, \quad \text{where} \quad \begin{matrix} P_{11} = \text{lower triangular } k \times k, \\ I = (n-k) \times (n-k). \end{matrix}$$

In the sequel we shall omit superscript k, it being understood that all matrices refer to stage k. Instead, superscripts will henceforth denote row numbers.

The source of trouble. Now ill-conditioning reveals itself by the fact that at some stage in the elimination procedure an entire row of S takes on very small values.

Consider the case where all elements of row S^i are very small. Elimination will then lead, at the present or some later stage, to division of this row by one of its very small current elements in order to provide a coefficient of value 1. This is equivalent to multiplication of row i by a very large number. Accumulated round-off errors are thereby greatly magnified. It is the high *relative* errors infecting this row that introduce the sudden substantial drop in accuracy, subsequently propagating to the entire solution vector. The situation could be saved if it were possible to obtain row S^i, which is the result of many computational steps, free of the high accumulated round-off errors. We shall show how to accomplish this with a comparatively small amount of labor.

The remedy. S^i is equivalent to the result of subtracting a linear combination of rows $A^1 \cdots A^k$ from A^i. We would first like to determine approximately the constants of this linear combination, indicating in what way S^i is obtainable directly from the given original matrix A.

From $A = LR$ (stage k understood) we obtain $R = L^{-1}A = PA$. We can therefore obtain S^i $(i > k)$ from

$$S^i \sim R^i = P^iA = (P^i_{21}, I^i)A = p_{i1}A^1 + \cdots + p_{ik}A^k + A^i.$$

Since the elements of the left-hand side are very small, this relation indicates that $p_{i1} \cdots p_{ik}$ are the negatives of the constants of linear combination of rows $A^1 \cdots A^k$ approximately furnishing A^i.

We therefore need the elements of P^i for $i > k$, i.e. of P^i_{21}. They are found by considering

$$PL = I, \qquad \text{i.e.,} \qquad \begin{pmatrix} P_{11} & 0 \\ P_{21} & I \end{pmatrix}\begin{pmatrix} L_{11} & 0 \\ L_{21} & I \end{pmatrix} = \begin{pmatrix} I & 0_{12} \\ 0_{21} & I \end{pmatrix}$$

and equating the expressions for block matrix 0_{21}. For row i ($> k$), this decomposition yields $P^i_{21}L_{11} + L^i_{21} = 0$, i.e.,

$$\begin{aligned} p_{i1}l_{11} + p_{i2}l_{21} + \cdots + p_{ik}l_{k1} &= -l_{i1}\,, \\ p_{i2}l_{22} + \cdots + p_{ik}l_{k2} &= -l_{i2}\,, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ p_{ik}l_{kk} &= -l_{ik}\,. \end{aligned}$$

The coefficients of this system are already available from preceding reduction steps. However, they bear accumulated round-off errors. Solution of this system would therefore only yield values p'_{ij} which approximate the true p_{ij}. Fortunately, that is all we need.

The preceding system, which is only of k-th order and triangular, is therefore solved recursively for the approximate solution $p'_{i1} \cdots p'_{ik}$. In a computer, the program for the "back solution" of the elimination procedure, already available within the machine, is effectively drawn upon for this purpose.

As shown above, the exact p_{ij}, applied to rows A^i, would yield the row S^i of very small elements, caused by cancellations in the most significant digits of each element. Instead, we take a linear combination with values p'_{ij} and replace the original matrix row A^i by

$$A'^i = p'_{i1}A^1 + \cdots + p'_{ik}A^k + A^i.$$

Similar cancellations of significant digits will again take place, leading to a row A'^i with small elements. This time, however, we shall take care to calculate with *double-precision*, furnishing small elements of A'^i that are *exact*.

Now we multiply through by a large number M to convert all elements to $O(1)$. Since the small elements were *exact*, no error is introduced or magnified by this step. The usual slump in accuracy has thus been completely prevented, maintaining the latter at its normal level.

The first k elements of the resulting row are now removed, and the row is thereby reduced to the form of R^i, by application of the elimination algorithm, utilizing the available rows $R^1 \cdots R^k$. It is in this reduction that the bulk of accumulation of round-off errors occurs. *Now*, however, the final R^i will have elements of $O(1)$, so that the effect of rounding errors will be no more appreciable than for all other normal rows.

Throughout k stages of reduction, row i ($> k$) has no effect on values in all other rows of R. Hence, apart from row i, the entire matrix R is retained as is.

We may look upon the reconditioning of matrix A in yet another light: When two matrices have their elements scaled such that corresponding row vectors have equal length (say all length 1), a comparison of their determinants provides an effective

comparison of "condition". Consider this done for the original matrix A (containing A^i) and for the modified A' (containing A'^i times the large number M). After complete reduction, we have $| A | = | L | \cdot | R | = \prod_{i=1}^{n} l_{ii}$. In the original system, a very small l_{ii} gave rise to a small $| A |$, indicating ill-condition. In the modified system, the element l'_{ii} in the identical position has been brought up to $O(1)$, carrying $| A' |$ to a comparatively high magnitude, which indicates favorable condition.

Summary. We summarize the auxiliary computations on encountering ill-conditioning: (1) recursive solution of a triangular linear system of order k; (2) double-precision recalculation of row A'^i, followed by a "left shift"; (3) elimination of the first k elements of row i.

It is noteworthy that removal of ill-conditioning requires the double-precision recalculation of only a single row and that intermediary results for all other rows are retained unaltered, thereby salvaging the bulk of computational effort.

Numerical example. We now illustrate the power of the present method by a numerical example. To simplify illustration, we shall desist from "positioning for size" for optimum divisors, but shall treat rows and columns in their original order.

Given the system

$$\begin{aligned}
1.32x_1 - 4.73x_2 + 5.39x_3 - 2.84x_4 + 3.97x_5 &= 1.33,\\
5.68x_1 - 6.25x_2 + 1.40x_3 + 7.02x_4 + 4.50x_5 &= -6.04,\\
1.93x_1 + 1.34x_2 - 2.16x_3 + 3.81x_4 - 2.62x_5 &= -1.75,\\
2.85x_1 + 3.09x_2 + 4.41x_3 + 2.36x_4 + 3.14x_5 &= 2.34,\\
4.32x_1 + 0.20x_2 + 4.69x_3 - 1.63x_4 - 5.39x_5 &= 4.50.
\end{aligned}$$

Suppose we stipulate that whenever, in any row, the element of maximum magnitude becomes less than 0.20, the entire row has fallen below an acceptable threshold of accuracy.

Applying straightforward elimination and rounding all figures to two decimal places, we obtain, after $k = 3$ stages, the system $R^{(3)}x = b^{(3)}$, i.e. $A = L^{(3)}R^{(3)}$, with the numerical values

$$(R^{(3)}, b^{(3)}) = \begin{bmatrix}
1 & -3.58 & 4.08 & -2.15 & 3.01 & 1.01\\
 & 1 & -1.55 & 1.37 & -0.89 & -0.84\\
 & & 1 & -1.21 & -0.39 & 1.17\\
 & & & 6.47 & 11.61 & -5.03\\
 & & & -0.08 & -0.01 & 0.02
\end{bmatrix},$$

$$L^{(3)} = \begin{bmatrix}
1.32 & & & & \\
5.68 & 14.08 & & & \\
1.93 & 8.25 & 2.76 & & \\
2.85 & 13.29 & 13.38 & 1 & \\
4.32 & 15.67 & 11.35 & 0 & 1
\end{bmatrix}.$$

The smallness of elements in row $i = 5$ now reveals the hitherto hidden fact that row 5 of A is almost a linear combination of the first $k = 3$ matrix rows. Writing L^T = transpose of the upper left-hand 3×3 submatrix L_{11} of $L^{(3)}$, we therefore solve the triangular system $L^T p^i + l^i = 0$, i.e.

$$1.32p_{51} + 5.68p_{52} + 1.93p_{53} = -4.32$$
$$14.08p_{52} + 8.25p_{53} = -15.67$$
$$2.76p_{53} = -11.35$$

still always rounding to two decimal places, yielding the approximate solution $p'_{51} = -2.86$, $p'^{1}_{52} = 1.30$, $p'_{53} = -4.11$. These values are utilized in replacing row 5 of the original augmented matrix (A, b) by the row

$$A'^{5} = p'_{51}A^1 + p'_{52}A^2 + p'_{53}A^3 + A^5$$

calculated to double-precision, i.e. the fifth equation is replaced by

$$-0.0035x_1 + 0.0954x_2 - 0.0278x_3 - 0.0407x_4 - 0.1260x_5 = 0.0367$$

We now multiply through by a comparatively large number, say $M = 100$, and reduce the resultant first three terms to zero by subtracting multiples of rows 1, 2 and 3 of the matrix $(R^{(3)}, b^{(3)})$. At the end of stage 3, row 5 of the augmented matrix $(R^{(3)}, b^{(3)})$ now becomes

$$(0, 0, 0, -2.26, 0.31; -2.47),$$

in which the numerical value of the element of largest magnitude is now far above the permissible minimum threshold of 0.20.

Completion of numerical solution yields final values for the unknowns which are compared in the table below. Deviations δ_i from the true values x_i as well as $\max |\delta_i|$ and $\sum |\delta_i|$ are likewise tabulated. The improvement in accuracy is indeed striking.

Comparison of numerical values

Quantity	Correct values	Usual method	Present method	Quantity	Usual method	Present method
x_1	−2	−0.30	−1.96	δ_1	−1.70	−0.04
x_2	0	0.39	0.00	δ_2	−0.39	0.00
x_3	2	0.77	1.95	δ_3	1.23	0.05
x_4	1	−0.23	0.96	δ_4	1.23	0.04
x_5	−1	−0.31	−0.97	δ_5	−0.69	−0.03
$\max \lvert\delta_i\rvert$	0	1.70	0.05			
$\sum \lvert\delta_i\rvert$	0	5.24	0.16			
p	0	87	3			

Relative percentage deviations $100 \mid \delta_i \mid / \mid x_i \mid$ for individual components become infinite or unduly large whenever $\mid x_i \mid = 0$ or very small. In order to summarize the

overall relative inaccuracy by a single number, avoiding at the same time any instability in the neighborhood of small $|x_i|$, we define as an effective measure of overall inaccuracy

$$p = \text{overall percentage error} = 100 \sum |\delta_i| / \sum |x_i|$$

Numerical values of overall measure p, shown in the above table, likewise compare very favorably.

Closeness of solution. Substituting the numerical solution into the given set of equations, we find that the residual vectors for the two sets of unknowns are both very small and have elements of comparable magnitude. How then can we say that our second set of x_i is any "better" than the first?

Let x = exact solution vector, x' = computed solution vector, $\delta = x - x'$ = error vector for the unknowns, $r = b - Ax'$ = residual vector for the computed solution. Substitution for x' in the expression for r yields $r = A\delta$.

In each case we obtain a small vector r, but this cannot serve as a valid criterion for closeness of solution. Our prime objective is the reduction not of r but of $\delta = x - x' = A^{-1}r$, and this objective has been successfully achieved.

TWO DIMENSIONAL COMPRESSIBLE SHEAR FLOW IN A CHANNEL*

BY
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1. Introduction. As far as the author is aware, no theoretical or numerical solution has been obtained for the two-dimensional, inviscid, adiabatic rotational flow of a compressible gas through a divergent channel such as shown in Fig. 1. Kramer and Stanitz [2] have obtained a relaxation solution for the incompressible shear flow in a 90° elbow. Mitchell [3] has derived a method for evaluating the non-isentropic rotational field downstream of the bow shock wave formed when a supersonic stream impinges on a blunt nosed obstacle.

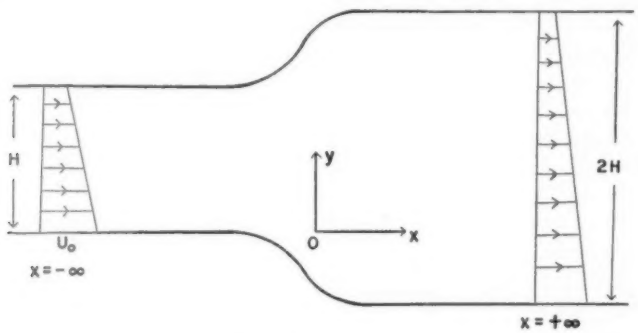

Fig. 1

In the present paper, it is intended to illustrate the principal physical features of the completely sub- and super-sonic adiabatic rotational flow through a channel, with particular reference to the effect of the divergence of the channel on the vorticity and the stream lines.

2. Fundamental equations and boundary conditions. In the two-dimensional steady motion of an inviscid, adiabatic, rotational compressible gas through a divergent channel (see Fig. 1), the stream function ψ is defined in cartesian coordinates by the equations

$$q_x = \rho^{-1} \frac{\partial \psi}{\partial y}, \qquad q_y = -\rho^{-1} \frac{\partial \psi}{\partial x},$$

where q_x and q_y are the cartesian velocity components, and ρ is the density. The vorticity ω is defined by

$$\omega = \frac{\partial q_y}{\partial x} - \frac{\partial q_x}{\partial y}, \tag{1}$$

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which, from Vaszonyi [4], gives the equation for ψ as

$$\frac{\partial}{\partial x}\left(\rho^{-1}\frac{\partial \psi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\rho^{-1}\frac{\partial \psi}{\partial y}\right)-\rho\frac{\partial h_s}{\partial \psi}=0, \tag{2}$$

where h_s is the stagnation enthalpy.

Bernoulli's equation for an adiabatic gas may be written as

$$\gamma(\gamma-1)^{-1}p/\rho+\tfrac{1}{2}q^2=h_s\,, \tag{3}$$

where p is the pressure, q the velocity, γ the ratio of specific heats, and $p\rho^{-\gamma}=$ constant. In an adiabatic rotational flow, h_s is necessarily a function of ψ.

As to the nature of the boundary conditions necessary for a unique solution to these equations in the general case, there is still some doubt. In the particular problem of the flow through a divergent channel with straight parallel sides far upstream and downstream of the constriction, the boundary conditions are more straightforward. The physical assumption is that at infinity upstream and downstream of the constriction the flow is parallel to the channel walls. Thus, taking the x-axis parallel and along the lower wall in the narrow section of the channel, the condition is $\partial\psi/\partial x=0$ at $x=\pm\infty$, the constriction being $x=0$. Since at $x=\pm\infty$, the flow is parallel, the pressure, and consequently the density, is constant across the channel. Accordingly, if h_s (or the vorticity) is given as a function of ψ, ψ is obtained from Eq. (2), and is thus known round a closed boundary, enclosing the field of flow, which in the elliptic case (subsonic flow) defines a unique solution over the complete field. Alternatively, if $\partial\psi/\partial x=0$, and ψ is given upstream at $x=-\infty$, and $\partial\psi/\partial x=0$ at $x=+\infty$, a solution is again defined.

Letting the suffix zero denote quantities at $x=-\infty$, it is convenient at this stage to introduce non-dimensionalising quantities p_0 , ρ_0 , H the channel width, and c_0 the speed of sound, where $c_0^2=\gamma p_0/\rho_0$. Thus, with only slight change in notation, ψ, p, ρ, q, ω, h_s , x and y, now stand for the non-dimensional quantities $\psi(\rho_0 c_0 H)^{-1}$, p/p_0 , ρ/ρ_0 , q/c_0 , $\omega H/c_0$, h_s/c_0^2 , x/H and y/H. In the following, it must be remembered that all quantities are non-dimensional.

In the specific problem, let the initial vorticity ω_0 be a constant, which from Eq. (1) gives

$$(q_x)_0=M_0-\omega_0 y, \tag{4}$$

where M_0 is the initial Mach number of the flow on the lower channel wall. The density is initially constant and equal to unity, which gives

$$\psi=M_0 y-\tfrac{1}{2}\omega_0 y^2. \tag{5}$$

From Eqs. (3), (4) and (5) the stagnation enthalpy h_s is given as a function of ψ by

$$h_s=(1/\gamma-1+\tfrac{1}{2}M_0^2)-\omega_0\psi. \tag{6}$$

The function $h_s(\psi)$ given by Eq. (6) remains unaltered throughout the flow, since h_s is constant along a streamline.

In the parallel flow region at $x=+\infty$, where the channel width is $2H$ (the general case rH is treated in a similar manner), $\partial\psi/\partial x=0$. Denoting conditions in this region by the suffix unity, Eq. (2) reduces to

$$\frac{\partial}{\partial y}\left(\rho^{-1}\frac{\partial \psi_1}{\partial y}\right) = -\rho_1\omega_0 ,$$

which on integration gives

$$\psi_1 = A + By - \tfrac{1}{2}\rho_1^2\omega_0 y^2,$$

where A and B are constants. On the lower channel wall at $y = -\frac{1}{2}$, $\psi_1 = 0$, and on the upper wall at $y = 3/2$, $\psi_1 = M_0 - \frac{1}{2}\omega_0$ [from Eq. (5)], which results in

$$\psi_1 = \tfrac{1}{8}(2M_0 - \omega_0 + 3\omega_0\rho_1^2) + \tfrac{1}{4}(2M_0 - \omega_0 + 2\omega_0\rho_1^2)y - \tfrac{1}{2}\omega_0\rho_1^2 y^2. \tag{7}$$

Thus, with the exception of ρ_1 , all quantities are known analytically in the parallel flow region at $x = +\infty$. Equation (3) must be satisfied at all points in the field, which in dimensionless form gives

$$(\gamma - 1)^{-1}\rho_1^{\gamma-1} + \frac{1}{2}\left(\rho_1^{-1}\frac{\partial \psi_1}{\partial y}\right)^2 = (\gamma - 1)^{-1} + \tfrac{1}{2}M_0^2 - \omega_0\psi_1 ,$$

where ψ_1 is given by Eq. (7). This last equation reduces to

$$\begin{aligned}\omega_0^2\rho_1^4 + 2(\gamma - 1)^{-1}\rho_1^{\gamma+1} + \rho_1^2[M_0\omega_0 - \tfrac{1}{2}\omega_0^2 - M_0^2 - 2(\gamma - 1)^{-1}] \\ + (\tfrac{1}{2}M_0 - \omega_0/4)^2 = 0.\end{aligned} \tag{8}$$

By Descarte's theorem, there cannot be more than two real positive roots of Eq. (8). It should be noted that Eq. (8) is not the general equation, but is that for a divergent two-dimensional channel whose area ratio is two. The general equation for any area ratio can be obtained quickly using the above method.

In Eq. (3), the stagnation enthalpy h_s is constant along a stream line (although it varies from one stream line to another), and so for any one such stream line

$$(\gamma - 1)^{-1} + \tfrac{1}{2}q_0^2 = (\gamma - 1)^{-1}\rho_1^{\gamma-1} + \tfrac{1}{2}q_1^2 .$$

From this last equation, it is seen that if $q_1 < q_0$, $\rho_1 > 1$, and if $q_1 > q_0$, $\rho_1 < 1$. Thus, the fact that $\rho_1 > 1$ for subsonic flow, and $\rho_1 < 1$ for supersonic flow is also the case when $\omega_0 > 0$. It is true for $\omega_0 = 0$, which is the well known irrotational flow through a divergent channel. Accordingly, the appropriate solution of Eq. (8) is chosen in each case: these are now considered separately.

3. Subsonic flow. In subsonic flow through the channel, the velocity at station unity is less than at station zero, and so ρ_1 is necessarily greater than unity. Thus, the requisite solution of Eq. (8) is that for $\rho_1 > 1$. With the restriction on ω_0 , given subsequently by Eq. (10), it can be quickly shown (by the method of signs) that the other real solution (> 0) of Eq. (8) is that in which $\rho_1 < 1$. From Eqs. (1) and (2), the final vorticity ω_1 is given as

$$\omega_1 = -\rho_1\frac{\partial h_s}{\partial \psi} = \rho_1\omega_0 , \tag{9}$$

which gives $\omega_1 > \omega_0$. That is, in the rotational flow of a compressible gas through a divergent channel, the vorticity will in general increase in value. Figure 2 illustrates how ω_1 varies with ω_0 , for several values of M_0 . There is a discontinuity in ω_1 , which will occur when $q_0 > 1$ and $q_0 < 1$ over some section of the channel near $x = -\infty$. In such a case the problem is of mixed type, and there is still some doubt as to the boundary

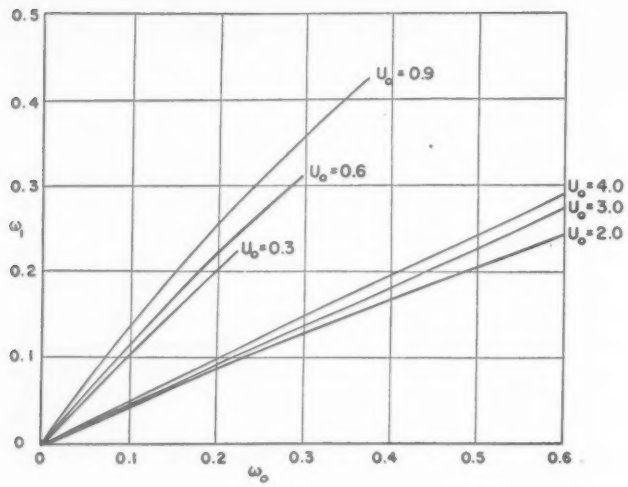

Fig. 2

conditions necessary for a solution to be defined. This difficulty does not arise when q_0 is everywhere greater than or less than unity.

In any physical flow, the velocity in the parallel flow regions must necessarily be in the direction of increasing x, that is, $q > 0$ at all points near $x = \pm\infty$. Thus, ω_0 must satisfy $\omega_0 < M_0$ from Eq. (4), and $\omega_0 < 2M_0\,(1 + 4\rho_1^2)^{-1}$ from Eq. (7). Together these give the following restriction on the initial vorticity ω_0 :

$$\omega_0 < 2M_0(1 + 4\rho_1^2)^{-1} < M_0\,. \tag{10}$$

Equation (10) ensures that there is no 'back' flow region (that is $q > 0$) in the field. The importance of this lies in the fact that if $\omega_0 < M_0$ ensuring that $q > 0$ near $x = -\infty$, the divergence of the channel may be such that the vorticity is increased sufficiently to make $q < 0$ at some points in the region near $x = +\infty$: that is ω_0 violates the first inequality in Eq. (10). Physically this could perhaps result in breakaway of the flow, the cause of which has always been entirely attributed to viscosity effects. It should be pointed out that Eq. (10) is again for an area ratio of two.

The effect of the divergence on the stream lines is marked, and is probably shown most profitably by the final deflection in the central stream line at $x = -\infty$, that is $(\psi_0)_{y=\frac{1}{2}} = (\frac{1}{2}M_0 - \omega_0/8)$. Substitution of this value for ψ_1 in Eq. (7), gives a quadratic equation, and the requisite value of y at $x = +\infty$ as the appropriate solution. The non-dimensional deflection δ of this stream line at $x = +\infty$ is given by Eq. (11), where y is the solution to the quadratic with the negative sign with the root bracket. The deflection δ must tend to zero as ω_0 tends to zero (irrotational flow), which necessitates the negative sign used in Eq. (11). Thus, the deflection is given by

$$\delta = y - 0.5 = (2\omega_0\rho_1^2)^{-1}[(M_0 - \tfrac{1}{2}\omega_0) - (M_0^2 + \omega_0^2/4 + 4\omega_0^2\rho_1^4 - \omega_0 M_0 - \omega_0^2\rho_1^2)^{1/2}], \tag{11}$$

from which it is seen that $\delta \to 0$ as $\omega_0 \to 0$. The root bracket in Eq. (11) may be written as $[(M_0 - \frac{1}{2}\omega_0)^2 + \omega_0^2\rho_1^2(4\rho_1^2 - 1)]^{\frac{1}{2}}$, which is greater than $(M_0 - \frac{1}{2}\omega_0)$ when $\rho_1 > 1$,

which is the subsonic case. Thus, from Eq. (11), $\delta < 0$ when the flow is subsonic, for any ω_0 and M_0 satisfying Eq. (10). The variation in δ with initial vorticity for several values of the initial velocity M_0, is illustrated in Fig. 3. One important feature of the flow arising from the figure is the fact that this deflection decreases with increase in the initial velocity for the same value of the vorticity. It should also be noted that the central stream line moves to a region of higher velocity at $x = +\infty$ than the central value in this region.

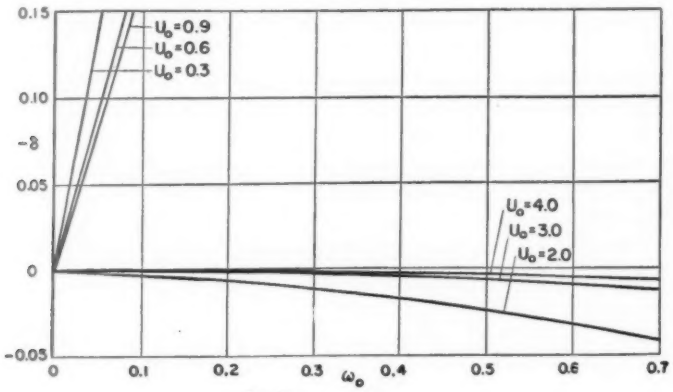

Fig. 3

In a problem of this type, which is elliptic, a simultaneous relaxation process can be carried out over the complete field of flow. The author has in fact evaluated such a problem, where the constriction was discontinuous, each wall having two right-angled corners. This relaxation solution was compared with a similar one for incompressible flow through the same channel, and it was found that in general the displacement effects were smaller in the compressible flow, but of the same sign.

4. Supersonic flow. In the completely supersonic flow through the channel, assuming that no shock waves are present, the differential equations are intrinsically different. This case is a hyperbolic problem, amenable to solution by characteristics, and conditions need only be given initially at $x = -\infty$. However, from physical considerations, it is assumed that the flow near $x = +\infty$ is parallel to the walls of the channel as before.

In this case the divergence of the channel increases the velocity, and the value of ρ_1 is the requisite solution of Eq. (8) satisfying $0 < \rho_1 < 1$. Accordingly, it follows from Eq. (9) that $\omega_1 < \omega_0$, and the restriction on ω_0 for completely supersonic flow at all points is, from Eq. (4)

$$\omega_0 < M_0 - 1. \tag{12}$$

Thus, the divergence decreases the vorticity, and the variation of ω_1 with ω_0 for several values of M_0 is shown in Fig. 2. In this case $q_0 > 1$ at all points.

The deflection in the central stream line is again given by Eq. (11), the negative sign being again used with the root bracket for the same reason as in Sec. 3. From Eq. (8), with ω_0 and M_0 (> 1) satisfying Eq. (12), it can be shown that ρ_1 is in fact less than 0.5 [this can also be seen immediately from Fig. 2 and Eq. (9)] and so the root exponent

in Eq. (11) becomes $[(M_0 - \frac{1}{2}\omega_0)^2 - \omega_0^2\rho_1^2(1 - 4\rho_1^2)]^{\frac{1}{2}}$, which, since $1 > 4\rho_1^2$, is less than $(M_0 - \frac{1}{2}\omega_0)$. Thus, Eq. (11) gives $\delta > 0$ for all ω_0 and M_0 (> 1) satisfying Eq. (12) in the completely supersonic flow (that is, where $\rho_1 < 1$). The variation of δ with ω_0 and M_0 is shown in Fig. 3. The essential difference between this case and the subsonic case, is that the central stream line at $x = -\infty$ is deflected towards a region of lower velocity at $x = +\infty$ than the central value at this station. Further, the magnitude of the deflection is very much smaller (in fact an order of magnitude different) in the supersonic case. One point of similarity is that the deflection decreases with increase in velocity.

In supersonic shear flow the effect of the vorticity on the deflection of the downstream stream lines is an order of magnitude different from that in the subsonic flow. This phenomenon substantiates the experimental work of Davies [1], who found that the deflection in the free stream of the stagnation stream line when a pitot tube is placed in a supersonic shear flow is opposite in sign and of a very much smaller magnitude from that obtained by Young and Maas [5] in the low subsonic case.

5. Discussion of results. By considering the differential equations (2) and (3), the variation in the vorticity due to the divergence in the channel is obtained, when the flow is, in one case, completely subsonic and, in the other, completely supersonic. It is shown that the difference between initial vorticities becomes smaller with increase in velocity in both cases, but in the subsonic case the final vorticity is greater than the initial value, the opposite being the case in the supersonic flow, as is shown in Fig. 2.

The effect of the divergence of the channel on the stream lines is characterised by consideration of the deflection in the initial central stream line, as is illustrated in Fig. 3. It is found that the deflection in the supersonic case is of an order of magnitude smaller and opposite in sign, from that found in the subsonic case. This result is in agreement with the state of affairs in the experimental work on supersonic flow past a pitot tube carried out by Davies.

One final remark is that there is a possibility of 'back' flow, which could result in breakaway of the flow in the subsonic rotational flow through a channel of this type. Restrictions on the initial rotation and channel width necessary for no 'back' flow to be present are derived.

The case of a divergent channel with general area ratio can be derived using the method described in this paper, with a proportional increase in algebraic labour.

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INTRINSIC FORM OF THE CHARACTERISTIC RELATIONS IN THE STEADY SUPERSONIC FLOW OF A COMPRESSIBLE FLUID*

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1. Introduction. The purpose of this paper is to express the characteristic relations for the steady, supersonic, three-dimensional, rotational and irrotational motion of a polytropic gas in intrinsic form and to apply these relations to the study of a class of space Beltrami motions. This class of motions is characterized by the fact that one family of the characteristic surfaces consists of ∞^1 parallel planes.

To obtain intrinsic forms of the characteristic relations, two techniques must be used. First, the equations of motion, continuity, and energy must be expressed in terms of a set of characteristic variables. Secondly, these must be reduced to intrinsic form by expressing the partial derivatives of the components of the velocity vector in terms of the directional derivatives of the magnitude of the velocity vector, q, and the sound speed, c, and the curvatures associated with the characteristic manifolds.

The class of fluid motions to be studied is characterized by the fact that one family of ∞^1 characteristic surfaces consists of planes, $z =$ constant. If the additional assumption is made that the bicharacteristics and their orthogonal trajectories determine two families of cylindrical surfaces with generators parallel to the z-axis, then it is shown that these surfaces are right circular cylinders with oz as axis; the bicharacteristics are concentric circles; the stream lines are helices along which q (or M, the Mach number) is constant.

2. The basic relation in terms of characteristic variables. Let x^i, $j = 1, 2, 3$, denote a Cartesian orthogonal coordinate system in Euclidean three-space and let us denote partial derivatives by the symbolism

$$\partial_i \equiv \frac{\partial}{\partial x^i}.$$

In a Cartesian orthogonal coordinate system, covariant and contravariant quantities are equivalent. However, in order to use the Einstein summation convention of summing on repeated lower and upper indices, we shall use these two equivalent quantities.

First, we consider the equations of continuity, motion, and energy in a Cartesian orthogonal coordinate system. Let ρ denote the specific density and v^i denote the velocity vector, then the equation of continuity is

$$v^i \, \partial_i \rho + \rho \, \partial_i v^i = 0. \tag{2.1}$$

For non-isentropic flows of a polytropic gas, the equations of motion may be written in the form [1]

$$\gamma \rho v^i \, \partial_i v_k + \partial_k (\rho c^2) = 0, \tag{2.2}$$

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where c is the local sound speed which may be defined in terms of the pressure, p, and the specific entropy S by

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_S,$$

and γ is the constant of the gas law

$$p = \rho^\gamma f(S).$$

In addition, we shall use as energy equation the Bernoulli relation

$$v^i \partial_i \left(\frac{c^2}{\gamma - 1} + \frac{q^2}{2}\right) = 0, \tag{2.3}$$

where $q^2 = v^i v_i$ is the magnitude squared of the velocity vector.

By definition, the characteristic manifolds of the system (2.1), (2.2), and (2.3) are those surfaces along which discontinuities in the derivatives, $\partial_i \rho$, $\partial_i c$, $\partial_i v_k$, can occur. Using the methods of a previous paper [2], these surfaces can be easily determined. However, these methods are not pertinent to the present problem; hence, the procedure which follows will be by analogy. By eliminating ρ between (2.1) and (2.2) and then by eliminating $\partial_i c^2$ in the resulting equation through use of (2.3), we obtain the following relation

$$(v^i v^k - c^2 g^{ik})\, \partial_i v_k = 0, \tag{2.4}$$

where g^{ik} represents the metric tensor. Thus, the *same basic equation* (2.4) is valid in the non-isentropic as well as in the isentropic case [3].

If the symmetric tensor a^{ik} is defined by

$$a^{ik} = v^i v^k - c^2 g^{ik}, \tag{2.5}$$

then for isentropic flows the characteristic manifolds of (2.4) are determined by

$$a^{ik} n_i n_k = 0, \tag{2.6}$$

where n_i is a unit vector orthogonal to one family of characteristic surfaces. It can be shown that (2.6) determines the characteristic manifolds in the non-isentropic case [2]. From the known theory of characteristic manifolds [4], it follows that

$$v^i = cn^i + bt^i, \tag{2.7}$$

where t^i is the unit tangent vector to the bicharacteristic curves and

$$b = (q^2 - c^2)^{1/2}. \tag{2.8}$$

We shall require that both b and c be positive. However, we shall not fix the *sense* of t^i, n^i. This means that we shall be able to use either nappe of the normal or bicharacteristic cones.

By use of (2.5), (2.7), we find that

$$a^{ik} n_k = bct^i, \tag{2.9}$$

$$a^{ik} t_k = bcn^i + (q^2 - 2c^2) t^i. \tag{2.10}$$

Secondly, we shall introduce a few basic relations from differential geometry [5].

Since n^i is a unit vector orthogonal to ∞^1 surfaces, we may write

$$\partial_i n_k = s_{ik} + n_i u_k , \qquad s_{ik}n^k = u_k n^k = 0, \tag{2.11}$$

where s_{ik} is the symmetric second fundamental tensor of the ∞^1 surfaces orthogonal to n_i and u_k is the curvature vector of the n_i congruence of curves. Further, let p_i denote a unit vector field which is orthogonal to both t_i and n_i so that p_i , t_i , and n_i form an ordered orthogonal right hand triple at each point. The curvature vector of the p_i congruence of curves will be denoted by w_i , and the curvature vector of the t_i congruence by m_i so that

$$\begin{aligned} w_i &= p^k \partial_k p_i . \\ m_i &= t^k \partial_k t_i . \end{aligned} \tag{2.12}$$

From the decomposition of the metric tensor

$$g^{ik} = t^i t^k + n^i n^k + p^i p^k,$$

we obtain the relation

$$g^{ik} \partial_i t_k = t^i t^k \partial_i t_k + n^i n^k \partial_i t_k + p^i p^k \partial_i t_k . \tag{2.13}$$

It is easily seen that

$$t^i t^k \partial_i t_k = 0, \qquad n^i n^k \partial_i t_k = -t_k u^k, \qquad p^i p^k \partial_i t_k = -t_k w^k. \tag{2.14}$$

Thus, we may express (2.13) in the form

$$g^{ik} \partial_i t_k = -t_k(u^k + w^k). \tag{2.15}$$

In this paragraph, the relations (2.7), (2.11), (2.15) will be used to express the *basic relation* (2.4) in terms of the rates of change of q and c with respect to displacements along t^i and n^i, and in terms of the curvatures s_{ik} , u_i , w_i . The following notation for directional derivatives will be used

$$\frac{\partial}{\partial t} \equiv t^i \partial_i , \qquad \frac{\partial}{\partial n} \equiv n^i \partial_i , \qquad \frac{\partial}{\partial p} \equiv p^i \partial_i .$$

Thus, $\partial/\partial t$ represents rate of change with respect to displacement along t^i, and similar interpretations are valid for $\partial/\partial n$, $\partial/\partial p$. By differentiation of (2.7), we find

$$\partial_i v_k = c \partial_i n_k + b \partial_i t_k + n_k \partial_i c + t_k \partial_i b. \tag{2.16}$$

Multiplying (2.16) by a^{ik} and using (2.9), (2.10), we obtain the relation

$$a^{ik} \partial_i v_k = c a^{ik} \partial_i n_k + b a^{ik} \partial_i t_k + cb\left(\frac{\partial c}{\partial t} + \frac{\partial b}{\partial n}\right) + (q^2 - 2c^2) \frac{\partial b}{\partial t}. \tag{2.17}$$

To evaluate the term $a^{ik}\partial_i n_k$ in equation (2.17), we use (2.5), (2.7) and find

$$a^{ik} = c^2(n^i n^k - g^{ik} - t^i t^k) + q^2 t^i t^k + cb(t^i n^k + n^i t^k). \tag{2.18}$$

Forming the scalar product of (2.11), (2.18), we obtain

$$a^{ik} \partial_i n_k = cbt^k u_k + b^2 s_{ik} t^i t^k - c^2 M^* \tag{2.19}$$

where M^* is the mean curvature of the characteristic surfaces

$$M^* = g^{ij} s_{ik} .$$

To determine the term $a^{ik}\partial_i t_k$ of equation (2.17), we use (2.18), (2.14), (2.15) and find

$$a^{ik}\,\partial_i t_k = -cbs_{ik}t^i t^k + c^2 t_k w^k. \tag{2.20}$$

If (2.19), (2.20) are substituted into (2.17) and the left hand side of the latter equation is equated to zero, we obtain the desired intrinsic form of the *basic relation* (2.4)

$$cb\left(\frac{\partial c}{\partial t} + \frac{\partial b}{\partial n}\right) + (q^2 - 2c^2)\,\frac{\partial b}{\partial t} + c^2 b t_k(u^k + w^k) - c^3 M^* = 0. \tag{2.21}$$

3. The intrinsic conditions for rotational (or irrotational) motion in terms of characteristic variables. If e^{ijk} denotes the permutation tensor, then the vorticity vector ω^j is defined by

$$\omega^j = e^{ijk}\,\partial_i v_k\,. \tag{3.1}$$

Since the *ordered* triad p_i , t_i , n_i forms a right hand system, the following formulae are obtained by forming the scalar products of (3.1) with the vectors t_i , n_i , p_i , respectively

$$\begin{aligned}
\omega^j t_j &= (n^i p^k - n^k p^i)\,\partial_i v_k\,, \\
\omega^j n_j &= (p^i t^k - p^k t^i)\,\partial_i v_k\,, \\
\omega^j p_j &= (t^i n^k - t^k n^i)\,\partial_i v_k\,.
\end{aligned} \tag{3.2}$$

A lengthy but direct computation by use of (2.11), (2.12), (2.16) shows that (3.2) may be written as

$$\begin{aligned}
\omega^j t_j &= cp^k u_k - \frac{\partial c}{\partial p} + b(n^i p^k - n^k p^i)\,\partial_i t_k\,, \\
\omega^j n_j &= \frac{\partial b}{\partial p} - bp^k m_k\,, \\
\omega^j p_j &= \frac{\partial c}{\partial t} - \frac{\partial b}{\partial n} - bs_{ik}t^i t^k - ct_k u^k.
\end{aligned} \tag{3.3}$$

The Eqs. (3.3) determine the intrinsic form of the conditions for irrotational motion.

Now, we consider a form of the equations of motion which contain the vorticity [6]

$$\partial_i h_0 - T\,\partial_i S = e_{jik}v^j\omega^k. \tag{3.4}$$

Here, T is the absolute temperature, S is the specific entropy, and h_0 is the stagnation enthalpy

$$h_0 = \frac{c^2}{\gamma - 1} + \frac{q^2}{2}\,, \tag{3.5}$$

where γ is the ratio of the specific heats of the polytropic gas. Since the stagnation enthalpy is constant along a stream line (see 2.3), with the aid of (2.7) we find that

$$c\,\frac{\partial h_0}{\partial n} + b\,\frac{\partial h_0}{\partial t} = 0 \tag{3.6}$$

and S satisfies a similar relation.

To express (3.4) in intrinsic form, the scalar product of this relation with the vectors t_i , n_i , and p_i will be found. Using (2.7), (3.1), and (3.3), the following formulae are

obtained

$$\frac{\partial h_0}{\partial t} - T\frac{\partial S}{\partial t} = c\left(\frac{\partial c}{\partial t} - \frac{\partial b}{\partial n} - bs_{ik}t^i t^k - ct^k u_k\right) \tag{3.7}$$

$$\frac{\partial h_0}{\partial n} - T\frac{\partial S}{\partial n} = -b\left(\frac{\partial c}{\partial t} - \frac{\partial b}{\partial n} - bs_{ik}t^i t^k - ct^k u_k\right) \tag{3.8}$$

$$\frac{\partial h_0}{\partial p} - T\frac{\partial S}{\partial p} = q\frac{\partial q}{\partial p} - (b^2 m_k + c^2 u_k)p^k - cbn^i p^k(\partial_i t_k - \partial_k t_i). \tag{3.9}$$

Evidently, (3.8) is a consequence of (3.6), (3.7).

The above equations can be described by saying that they form a system consisting of three equations (3.7), (3.9), and (2.21) in the unknowns

$$\frac{\partial q}{\partial t}, \quad \frac{\partial q}{\partial n}, \quad \frac{\partial q}{\partial p},$$

where h_0 and S are prescribed functions which are constant along a stream line. The sound speed, c, is determined as a function of the magnitude of the velocity, q, by the Bernoulli relation (3.5). Thus, the roles of c and q may be interchanged. In the applications, the stream lines are unknown. Hence, the problem is to determine the functions h_0 , S, and q so that two relations of the type (3.6) are satisfied (one equation in the derivatives of h_0 and the other in the derivatives of S) and also Eqs. (3.7), (3.9) and (2.21) are valid.

The curvature term,

$$K = n^i p^k(\partial_i t_k - \partial_k t_i), \tag{3.10}$$

in the right hand side of (3.9) will be briefly considered. Through use of (2.11), this term may be written as

$$K = -(n^i t^k\, \partial_i p_k - t^i p^k\, \partial_k n_i) = -n^i t^k\, \partial_i p_k + s_{ki}t^i p^k. \tag{3.11}$$

If the unit vector field p_i is orthogonal to ∞^1 surfaces, as is the case in plane and axial-symmetric flow, then

$$\partial_i p_k = r_{ik} + p_i w_k\,, \qquad p^k r_{ik} = p^k w_k = 0,$$

where r_{ik} is the symmetric second fundamental tensor of these surfaces and w_k is the curvature vector of the p_i congruence (see 2.12). In this case, (3.11) reduces to

$$K = -r_{ik}n^i t^k + s_{ik}t^i p^k. \tag{3.12}$$

For the special cases of plane and axial-symmetric flow, K vanishes. In both cases, p_i is orthogonal to ∞^1 planes and hence $r_{ik} = 0$ and p_i , t_i are in the principal directions of the surfaces orthogonal to n_i [7]. The ∞^2 planes are parallel in plane flow; they intersect in a common line in axially symmetric flow.

4. Canonical form of the system (2.21), (3.7), and (3.9). In this section the equations (2.21), (3.7), and (3.9) will be written so that one equation contains only the directional derivative, $\partial/\partial t$, a second equation contains only the directional derivative $\partial/\partial n$, and the final equation contains only the directional derivative, $\partial/\partial p$.

If (3.7) is multiplied by b and the resulting equation is added to (2.21) the following relation is obtained.

$$b^2 \frac{\partial}{\partial t} \frac{q^2}{b} = -c^2 b t_k w^k + b^2 c s_{ik} t^i t^k + c^3 M^* + b\left(\frac{\partial h_0}{\partial t} - T \frac{\partial S}{\partial t}\right). \tag{4.1}$$

For $q > c$, the equation (4.1) is the first of the desired canonical relations.

To determine the second relation of desired canonical system, we multiply (3.7) by b and subtract from (2.21). We find that

$$2cb \frac{\partial b}{\partial n} + (q^2 - 2c^2) \frac{\partial b}{\partial t} + c^2 b t_k (2u^k + w^k) - c^3 M^* + cb^2 s_{ik} t^i t^k = -b\left(\frac{\partial h_0}{\partial t} - T \frac{\partial S}{\partial t}\right). \tag{4.2}$$

To evaluate the second term of the left hand side of (4.2), we use (3.5), (3.7) and obtain

$$(\gamma - 1) b \frac{\partial b}{\partial t} = -2 \frac{\partial h_0}{\partial t} + (\gamma + 1)\left[T \frac{\partial S}{\partial t} - c \frac{\partial b}{\partial n} - cbs_{ik} t^i t^k - c^2 t_k u^k\right]. \tag{4.3}$$

Converting $\partial h_0/\partial t$, $\partial S/\partial t$ into $\partial h_0/\partial n$, $\partial S/\partial n$ by use of (3.6) we find that (4.3) reduces to

$$(\gamma - 1) b^2 \frac{\partial b}{\partial t} = 2c \frac{\partial h_0}{\partial n} - (\gamma + 1)\left[cT \frac{\partial S}{\partial n} + bc \frac{\partial b}{\partial n} + cb^2 s_{ik} t^i t^k + c^2 b t_k u^k\right]. \tag{4.4}$$

Substituting (4.4) into (4.2) and converting $\partial h_0/\partial t$, $\partial S/\partial t$ of (4.2) into $\partial h_0/\partial n$, $\partial S/\partial n$ by use of (3.6), we obtain the second canonical relation for $q > c$.

$$\begin{aligned} b[(\gamma - 3)q^2 + 4c^2] \frac{\partial b}{\partial n} &= [2q^2 - (\gamma + 3)c^2] b^2 s_{ik} t^i t^k + (\gamma - 1) b^2 c^2 M^* \\ &\quad - cb[(\gamma - 3)q^2 + 4c^2] t_k u^k - (\gamma - 1) b^3 c t_k w^k \\ &\quad + [(\gamma - 3)q^2 - (\gamma - 5)c^2] \frac{\partial h_0}{\partial n} + [2q^2 - (\gamma + 3)c^2] T \frac{\partial S}{\partial n}. \end{aligned} \tag{4.5}$$

Evidently, (3.9) is the third equation of the desired canonical system. This equation can be written in the form

$$q \frac{\partial q}{\partial p} = \frac{\partial h_0}{\partial p} - T \frac{\partial S}{\partial p} + (b^2 m_k + c^2 u_k) p^k + cbK, \tag{4.6}$$

where K is the curvature defined in (3.10).

5. A class of space flows. For plane flows, the two families of characteristic surfaces can be chosen to be right cylinders with parallel generators which are perpendicular to the plane of the flow. Here, we consider *the case where one family of characteristic surfaces are parallel planes.* We shall show that this condition defines a family of space flows.

We recall that in Sec. 3, we assumed that the ordered triad of unit vectors p_i , t_i , n_i forms a right hand system. If the characteristic planes are assumed to be perpendicular to the z-axis of an x, y, z Cartesian orthogonal coordinate system, then n_i can be chosen to be sensed either in the positive z-direction or in the negative z-direction. In the first case, the ordered pair, p_i , t_i forms a right hand system when considered as part of the ordered triple p_i , t_i , and the positive z-direction. The second case leads to nothing new. Hence only the first case will be considered.

In any plane orthogonal to the z-axis, we introduce two families of parameter curves: α = variable along t_i , β = variable along p_i . Arc length will be positive when measured

in the directions of t_i , p_i , respectively. Every family of ∞^1 curves, α = variable (z = constant, β = constant), generates a surface. Any surface β = constant will be chosen so that it consists of ∞^1 curves, α = variable, lying in ∞^1 planes, z = constant. We choose the surfaces, α = constant, in a similar manner. The third family of surfaces in our new coordinate system will be the planes, $z = \delta$ = constant. The transformation equations relating α, β, δ to x, y, z are of the form

$$x = x(\alpha, \beta, \delta), \qquad y = y(\alpha, \beta, \delta), \qquad z = \delta. \tag{5.1}$$

In any plane, δ = constant, the arc length element is

$$ds^2 = (Ad\alpha)^2 + (Bd\beta)^2, \tag{5.2}$$

where A, B are functions of α, β, δ. If $\theta(\alpha, \beta, \delta)$ denotes the angle between the x-axis and t_i then the unit vectors t_i , p_i have Cartesian orthogonal components

$$\begin{aligned} t_i &: (\cos\theta, \sin\theta, 0), \\ p_i &: (\sin\theta, -\cos\theta, 0). \end{aligned} \tag{5.3}$$

Thus, we find that

$$\begin{aligned} \frac{\partial x}{\partial \alpha} &= A\cos\theta, \qquad \frac{\partial y}{\partial \alpha} = A\sin\theta, \\ \frac{\partial x}{\partial \beta} &= B\sin\theta, \qquad \frac{\partial y}{\partial \beta} = -B\cos\theta. \end{aligned} \tag{5.4}$$

In the case of curves lying in the planes, z = constant, the sense of the principal normal vector is chosen so that the ordered triple consisting of the tangent vector, the principal normal vector, and the positive z-direction form a right hand system. The curvature may be positive or negative. By use of the Frenet formula for plane curves and the fact that the principal normals of the p_k , t_k congruences lie along t_k , $-p_k$, respectively, we find

$$\begin{aligned} p^i\,\partial_i p_k &= w_k = k' t_k\,, \\ t^i\,\partial_i t_k &= m_k = -k p_k\,, \end{aligned} \tag{5.5}$$

where k' and k are the curvatures of the curves, β = variable and α = variable, respectively. From the equations (5.2), (5.5), it follows that

$$k' = \frac{t^i}{B}\frac{\partial p_i}{\partial \beta}\,, \qquad -k = \frac{p^i}{A}\frac{\partial t_i}{\partial \alpha}. \tag{5.6}$$

Substituting (5.3) into (5.6), we find that

$$k' = \frac{1}{B}\frac{\partial \theta}{\partial \beta}\,, \qquad k = \frac{1}{A}\frac{\partial \theta}{\partial \alpha}.$$

Again, we note that the integrability conditions of (5.4) are

$$\frac{\partial \theta}{\partial \alpha} = \frac{1}{B}\frac{\partial A}{\partial \beta}\,, \qquad \frac{\partial \theta}{\partial \beta} = -\frac{1}{A}\frac{\partial B}{\partial \alpha}. \tag{5.7}$$

The relations (5.5), (5.7), lead to the useful results

$$w_k t^k = k' = -\frac{1}{AB}\frac{\partial B}{\partial \alpha}, \qquad m_k p^k = -k = -\frac{1}{AB}\frac{\partial A}{\partial \beta}. \tag{5.8}$$

By use of (5.4), we see that the matrix of the transformation (5.1) is

$$\begin{bmatrix} \frac{\partial x}{\partial \alpha} & \frac{\partial y}{\partial \alpha} & \frac{\partial z}{\partial \alpha} \\ \frac{\partial x}{\partial \beta} & \frac{\partial y}{\partial \beta} & \frac{\partial z}{\partial \beta} \\ \frac{\partial x}{\partial \delta} & \frac{\partial y}{\partial \delta} & \frac{\partial z}{\partial \delta} \end{bmatrix} = \begin{bmatrix} A\cos\theta & A\sin\theta & 0 \\ B\sin\theta & -B\cos\theta & 0 \\ \frac{\partial x}{\partial \delta} & \frac{\partial y}{\partial \delta} & 1 \end{bmatrix} \tag{5.9}$$

Since the matrix of $\partial\alpha/\partial x$ etc. consists of the reduced cofactors of the determinant of the matrix (5.9), we find

$$\begin{bmatrix} \frac{\partial \alpha}{\partial x} & \frac{\partial \beta}{\partial x} & \frac{\partial \delta}{\partial x} \\ \frac{\partial \alpha}{\partial y} & \frac{\partial \beta}{\partial y} & \frac{\partial \delta}{\partial y} \\ \frac{\partial \alpha}{\partial z} & \frac{\partial \beta}{\partial z} & \frac{\partial \delta}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\cos\theta}{A} & \frac{\sin\theta}{B} & 0 \\ \frac{\sin\theta}{A} & -\frac{\cos\theta}{B} & 0 \\ -a_{13} & -a_{23} & 1 \end{bmatrix}, \tag{5.10}$$

where

$$a_{13} = \frac{\sin\theta}{A}\frac{\partial y}{\partial \delta} + \frac{\cos\theta}{A}\frac{\partial x}{\partial \delta}, \qquad a_{23} = -\frac{\cos\theta}{B}\frac{\partial y}{\partial \delta} + \frac{\sin\theta}{B}\frac{\partial x}{\partial \delta}. \tag{5.11}$$

Thus, we find by use of the chain rule and (5.10)

$$\frac{\partial}{\partial z}\Big)_{x,y} = -a_{13}\frac{\partial}{\partial \alpha} - a_{23}\frac{\partial}{\partial \beta} + \frac{\partial}{\partial \delta}. \tag{5.12}$$

The significance of the coefficients a_{13}, a_{23} can be seen from the following arguments. By use of (5.9), (5.11), we find that

$$a_{13} = \frac{1}{A^2}\left(\frac{\partial x}{\partial \alpha}\frac{\partial x}{\partial \delta} + \frac{\partial y}{\partial \alpha}\frac{\partial y}{\partial \delta} + (0)(1)\right), \qquad a_{23} = \frac{1}{B^2}\left(\frac{\partial x}{\partial \beta}\frac{\partial x}{\partial \delta} + \frac{\partial y}{\partial \beta}\frac{\partial y}{\partial \delta} + (0)(1)\right). \tag{5.13}$$

If $\cos(\alpha, \delta)$ denotes the cosine of the angle between the curves, β = constant, δ = constant, and α = constant, β = constant and $\cos(\beta, \delta)$ denotes the cosine of the angle between the curves, α = constant, δ = constant and α = constant, β = variable then

$$Aa_{13} = \cos(\alpha, \delta), \qquad Ba_{23} = \cos(\beta, \delta). \tag{5.14}$$

Again, we note that if we solve (5.11) for $\partial y/\partial\delta$, $\partial x/\partial\delta$, we obtain

$$\begin{aligned}\frac{\partial x}{\partial \delta} &= Ba_{23}\sin\theta + Aa_{13}\cos\theta,\\ \frac{\partial y}{\partial \delta} &= Aa_{13}\sin\theta - Ba_{23}\cos\theta.\end{aligned} \tag{5.15}$$

From (5.4), (5.15), we obtain a set of integrability conditions which must be added to the equations (5.7). These are

$$\frac{\partial}{\partial\alpha}(Ba_{23}) - Aa_{13}\frac{\partial\theta}{\partial\alpha} = -A\frac{\partial\theta}{\partial\delta}, \tag{5.16}$$

$$\frac{\partial}{\partial\alpha}(Aa_{13}) + Ba_{23}\frac{\partial\theta}{\partial\alpha} = \frac{\partial A}{\partial\delta}, \tag{5.17}$$

$$\frac{\partial}{\partial\beta}(Ba_{23}) - Aa_{13}\frac{\partial\theta}{\partial\beta} = \frac{\partial B}{\partial\delta}, \tag{5.18}$$

$$\frac{\partial}{\partial\beta}(Aa_{13}) + Ba_{23}\frac{\partial\theta}{\partial\beta} = B\frac{\partial\theta}{\partial\delta}. \tag{5.19}$$

From (5.14), we see that if the angles between the coordinate lines, α = variable and δ = variable, β = variable and δ = variable, are $\pi/2$, then $Aa_{13} = Ba_{23} = 0$. The above relations show that in this case, θ, A, B are all independent of δ (as is to be expected). The relations (5.7), (5.16) through (5.19) imply that the *Riemann tensor vanishes in the α, β, δ system.*

To determine formulas (4.1), (4.5), (4.6), (3.6) in terms of the *congruences* t^i, p^i, n^i, we must evaluate the curvature terms. Since, the n^i congruence consists of straight lines parallel to the z-axis, the curvature vector of these curves, u_k, vanishes. Further, the second fundamental tensor of the planes (z = constant) is $s_{jk} = 0$; also, the mean curvature, M^* of these planes vanishes. Again, we consider the curvature, K, of (3.10)

$$K = n^i p^k(\partial_i t_k - \partial_k t_i).$$

Since the vector t_i lies in the plane, z = constant

$$n^i p^k\,\partial_k t_i = 0.$$

Further, $n^i\partial_i t_k$ represents the directional derivative of t_k in the z-direction. From geometric considerations, or use of (5.3), it follows that

$$n^i\,\partial_i t_k = -\frac{\partial\theta}{\partial z}\Big)_{x,y} p_k\,.$$

By use of the above results and (5.12), the formula for K becomes

$$K = \left(a_{13}\frac{\partial\theta}{\partial\alpha} + a_{23}\frac{\partial\theta}{\partial\beta} - \frac{\partial\theta}{\partial\delta}\right). \tag{5.20}$$

Finally, the directional derivatives $\partial/\partial t$, $\partial/\partial n$, $\partial/\partial p$ become

$$\frac{\partial}{\partial t} = A^{-1}\frac{\partial}{\partial \alpha}, \qquad \frac{\partial}{\partial p} = B^{-1}\frac{\partial}{\partial \beta}$$
$$\frac{\partial}{\partial n} = \frac{\partial}{\partial z}\Big|_{x,y} = -\left(a_{13}\frac{\partial}{\partial \alpha} + a_{23}\frac{\partial}{\partial \beta} - \frac{\partial}{\partial \delta}\right). \tag{5.21}$$

Now, we express the system (4.1), (4.5), (4.6), (3.6) in terms of the metric quantities A, B, a_{12}, a_{13}, θ, and the partial derivatives $\partial/\partial\alpha$, $\partial/\partial\beta$, $\partial/\partial\delta$. By use of (5.8), (5.21), we find that (4.1) becomes

$$b\frac{\partial}{\partial \alpha}\frac{q^2}{b} = c^2\frac{\partial}{\partial \alpha}\ln B + \frac{\partial h_0}{\partial \alpha} - T\frac{\partial S}{\partial \alpha}. \tag{5.22}$$

Further, by use of (5.21), (5.8), relation (4.5) reduces to

$$\begin{aligned} -b[(\gamma-3)q^2 + 4c^2]&\left[a_{13}\frac{\partial b}{\partial \alpha} + a_{23}\frac{\partial b}{\partial \beta} - \frac{\partial b}{\partial \delta}\right] \\ &= (\gamma-1)b^3cA^{-1}\frac{\partial}{\partial \alpha}\ln B - [(\gamma-3)q^2 - (\gamma-5)c^2] \\ &\cdot\left[a_{13}\frac{\partial h_0}{\partial \alpha} + a_{23}\frac{\partial h_0}{\partial \beta} - \frac{\partial h_0}{\partial \delta}\right] - T[2q^2 - (\gamma+3)c^2] \\ &\cdot\left[a_{13}\frac{\partial S}{\partial \alpha} + a_{23}\frac{\partial S}{\partial \beta} - \frac{\partial S}{\partial \delta}\right]. \end{aligned} \tag{5.23}$$

By use of (5.20), (5.21), (5.8), we find that (4.6) becomes

$$q\frac{\partial q}{\partial \beta} = \frac{\partial h_0}{\partial \beta} - T\frac{\partial S}{\partial \beta} - b^2\frac{\partial}{\partial \beta}\ln A + cbB\left[a_{13}\frac{\partial \theta}{\partial \alpha} + a_{23}\frac{\partial \theta}{\partial \beta} - \frac{\partial \theta}{\partial \delta}\right]. \tag{5.24}$$

Finally, (3.6) reduces to

$$(-ca_{13} + bA^{-1})\frac{\partial h_0}{\partial \alpha} - ca_{23}\frac{\partial h_0}{\partial \beta} + c\frac{\partial h_0}{\partial \delta} = 0. \tag{5.25}$$

From (5.25), it follows that the differential equations of the stream lines are

$$\frac{Ad\alpha}{-Aa_{13} + (M^2-1)^{1/2}} = \frac{d\beta}{-a_{23}} = d\delta, \tag{5.26}$$

where M is the Mach number, q/c. Since $Ad\alpha$, $Bd\beta$, $d\delta$ are the arc length elements along the coordinate lines and Aa_{13}, Ba_{23} are cos (α, δ), cos (β, δ), it follows from (5.26) that in general the stream lines are space curves.

6. A family of isentropic Beltrami flows with plane characteristic surfaces (helical stream lines, concentric circular bicharacteristics). We shall consider those flows of Sec. 5 for which

$$h_0 = \text{constant}, \quad S = \text{constant}, \quad a_{13} = a_{23} = 0. \tag{6.1}$$

From (3.4), it follows that in any isentropic flow for which h_0 is constant, the vorticity vector is parallel to the veloctiy vector. Such flows are called Beltrami flows and are more general than irrotational flows. As noted earlier, (5.16) through (5.19) imply that

$$\frac{\partial A}{\partial \delta} = \frac{\partial B}{\partial \delta} = \frac{\partial \theta}{\partial \delta} = 0, \tag{6.2}$$

that is, the curves, δ = variable, are orthogonal to the planes, z = constant. Stated in other words, *every family of* ∞^1 *bicharacteristics* (δ = *variable*, α = *variable*) *forms surfaces with straight line generators parallel to the z-axis; also, every family of* ∞^1 *orthogonal trajectories of the bicharacteristics* (δ = *variable*, β = *variable*) *forms similar surfaces.* From (5.23), (6.1), (6.2), we see by use of a simple argument that

$$\frac{\partial q}{\partial \delta} = \frac{\partial B}{\partial \alpha} = 0. \tag{6.3}$$

From (6.2), (6.3), we see that, $B = 1$, by proper choice of scale factor along the β = variable curves. Further, (5.7) implies that the orthogonal trajectories of the bicharacteristics are straight lines or $k' = 0$, $\theta = \theta(\alpha)$ and

$$A = \beta\theta' + f(\alpha), \tag{6.4}$$

where $\theta' = d\theta/d\alpha$ and $f(\alpha)$ is an arbitrary function of α. Returning to (5.22), (5.23), (5.24), we see that (6.1), (6.2) lead to

$$q = q(\beta), \tag{6.5}$$

$$\frac{M^2}{2(M^2 - 1)} \frac{d}{d\beta} \ln \frac{M^2}{2 + (\gamma - 1)M^2} = -\frac{\partial}{\partial \beta} \ln A, \tag{6.6}$$

where $M = q/c$ is the Mach number. Since M is a function of only β, we see that in (6.4)

$$\theta(\alpha) = c_1\alpha, \qquad f(\alpha) = c_2c_1, \tag{6.7}$$

where c_1, c_2 are arbitrary constants. Integrating (6.6) and choosing a proper scale factor along the α = variable curves, we obtain

$$\frac{2 + (\gamma - 1)M^2}{M^2 - 1} = A^{(\gamma+1)}. \tag{6.8}$$

To interpret (6.8), we must determine the significance of A. From (6.7), (6.4) we find

$$A = c_1(\beta + c_2).$$

Since, $B = 1$, we obtain from (5.4)

$$x = (\beta + c_2) \sin \theta, \qquad y = -(\beta + c_2) \cos \theta. \tag{6.9}$$

From (6.9), we find that, $c_2 + \beta = r$, where r is the radial distance variable in polar coordinates. If we introduce the polar angle $\phi = \theta - \pi/2$ into (6.9) then

$$x = r \cos \phi, \qquad y = r \sin \phi. \tag{6.10}$$

Thus, (6.8), may be written as

$$\frac{2 + (\gamma - 1)M^2}{M^2 - 1} = Cr^{(\gamma+1)}, \tag{6.11}$$

where C is a constant. It is easily shown from (6.11) that for $\gamma > -1$, M decreases as r increases, $M = 1$ as r approaches infinity, M approaches infinity when $r = [(\gamma - 1)/C]^{1/(\gamma+1)}$.

From (6.9), we see that the bicharacteristics, $r = c_2 + \beta =$ constant, are concentric circles. By integrating (5.26), we obtain

$$\phi = \frac{(M^2 - 1)^{1/2}}{r} z + c_3 , \qquad r = \text{constant}, \tag{6.12}$$

where c_3 is an arbitrary constant. Thus, the stream lines are helices and cut the generators of the cylinder, $r =$ constant, in the constant Mach angle ψ, where

$$\tan \psi = (M^2 - 1)^{1/2}.$$

By use of (3.3), and the relations (see 5.5)

$$K = u_k = s_{ik} = \frac{\partial c}{\partial t} = \frac{\partial b}{\partial n} = 0, \qquad p^k m_k = -\frac{1}{r} ,$$

we find that

$$\omega^i = -\frac{dc}{dr} t^i + \left(\frac{db}{dr} + \frac{b}{r}\right) n^i . \tag{6.13}$$

Thus, the present class of flows are rotational flows. Evidently q is constant along each stream line [8].

By assigning various appropriate values to a_{13} , a_{23} , other flows may be obtained.

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WAVES PRODUCED BY A PULSATING SOURCE TRAVELLING BENEATH A FREE SURFACE*

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Summary. The propagation of surface waves induced by a pulsating source travelling beneath a free surface is investigated by introducing Rayleigh's dissipative force. By defining a disturbance velocity potential $R\varphi(x, y) \exp(i\omega t)$, it is shown that $\varphi(x, y)$ has to satisfy throughout the flow field not only the Laplace equation, but also a differential equation resulting from the analytic extension of the free surface boundary condition. The solution is obtained in the form

$$\varphi(x, y) = f(z) + g(\bar{z}), \qquad z = x + iy,$$

i.e., $\varphi(x, y)$ is a complex harmonic function of x, y but is not an analytic function of z. The wave propagation is found to depend on a parameter τ, which is the ratio of the pulsation frequency ω of the source strength to the fundamental frequency ω_0 of the surface wave produced by a constant strength source travelling at the same speed. The case $\tau = 0$, corresponding to the travelling source of constant strength, does give a single undamped harmonic wave train on the downstream side, of wave length $2\pi c^2/g$, (or frequency ω_0), as is expected. For $0 < \tau < 1/4$, there are four, and for $\tau > 1/4$, two undamped harmonic wave trains of different wave lengths on the downstream side. It is further observed that there exists a critical frequency at $\tau = 1/4$, at which frequency resonance phenomena occur. Thus violent disturbance is predicted at $\tau = 1/4$ by the present analysis. No disturbance is found to propagate infinitely upstream. This result evidently justifies the imposition of an asymptotic upstream condition of "vanishing disturbance at infinity" to replace the effect of the dissipative force in rendering the solution unique.[1]

1. Velocity potential. Consider a two dimensional incompressible fluid of infinite depth and extent, beneath whose free surface a point source is travelling at a constant speed c. In an (x, y) coordinate system that moves with the source, and under the hypothesis that the resulting fluid motion is irrotational, one can define an absolute disturbance velocity potential $\Phi(x, y, t)$ through the differential equation

$$\Phi_{xx} + \Phi_{yy} = 0 \tag{1}$$

and the boundary condition over the free surface $Y(y = 0)$

$$\Phi_t - c\Phi_x = -gY,$$

$$Y_t - cY_x = \Phi_y,$$

i.e.

$$\Phi_{tt} - 2c\Phi_{tx} + c^2\Phi_{xx} + g\Phi_y = 0. \tag{2}$$

When the strength of the travelling source fluctuates harmonically, we may write the velocity potential in the form:

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$$\Phi(x, y, t) = \Re\varphi(x, y) \exp(i\omega t) \tag{3}$$

and obtain the following differential system for φ:

$$\varphi_{xx} + \varphi_{yy} = 0, \tag{4}$$

$$\nu\varphi_y - 2i\tau\nu\varphi_x - \nu^2\varphi + \tau^2\varphi_{xx} = 0, \qquad y = 0, \tag{5}$$

where

$$\tau = \omega c/g, \qquad \nu = \omega^2/g.$$

It is a well known fact that the solution of this problem is indeterminate, as free harmonic surface wave trains of appropriate wave lengths can be added without disturbing either the differential equation or the boundary condition. This indeterminacy is removed by introducing an artificial dissipative force $-\mu \nabla \Phi$ and then letting μ approach zero in the result. This artifice was first introduced by Rayleigh[2], and has since been successfully employed by various investigators[3,4,5] for studying surface waves set up by moving sources and sinks of constant strengths.

By adopting this scheme, the differential equations (1) and (4) are not affected, but the boundary conditions (2) and (5) are replaced by

$$\Phi_{tt} - 2c\Phi_{tx} + c^2\Phi_{xx} + \mu\Phi_t - \mu c\Phi_x + g\Phi_y = 0, \tag{6}$$

$$\nu\varphi_y - 2i\tau\nu(1 - i\beta)\varphi_x - \nu^2(1 - 2i\beta)\varphi + \tau^2\varphi_{xx} = 0, \tag{7}$$

respectively for $y = 0$, where $\beta = \mu/2\omega$.

Let us now write the spatial part of our velocity potential (with moving source located at $\zeta = \xi + i\eta$) in the form

$$\varphi = \ln|z - \zeta| - \ln|z - \bar{\zeta}| + G(x, y; \xi, \eta), \qquad z = x + iy, \tag{8}$$

where $G(x, y; \xi, \eta)$ is harmonic on the lower half plane. Noting that on $y = 0$,

$$\begin{aligned} &\frac{\partial}{\partial y} \ln|z - \zeta| = -\frac{\partial}{\partial y} \ln|z - \bar{\zeta}| = \mathcal{I}(z - \bar{\zeta})^{-1}, \\ &\frac{\partial^n}{\partial x^n} \ln|z - \zeta| = \frac{\partial^n}{\partial x^n} \ln|z - \bar{\zeta}|, \end{aligned} \tag{9}$$

and substituting (8) into (7), we obtain the following differential equation for G on $y = 0$:

$$\nu G_y - 2i\tau\nu(1 - i\beta)G_x - \nu^2(1 - 2i\beta)G + \tau^2 G_{xx} = 2\nu \frac{\partial}{\partial y} \ln|z - \bar{\zeta}|. \tag{10}$$

As both sides of (10) involve functions that are harmonic in the lower half plane, (10) may be required to hold for the entire lower half plane by analytic extension. It can indeed easily be shown that a general solution of the Laplace equation is given by $f(z) + g(\bar{z})$.

Now we have the well known relation

$$(z - \bar{\zeta})^{-1} = i \int_0^\infty \exp[-i\kappa(z - \bar{\zeta})]\, d\kappa, \qquad y + \eta < 0 \tag{11}$$

from which the following expression can readily be deduced:

$$2\nu \frac{\partial}{\partial y} \ln | z - \zeta | = -\nu \int_0^\infty \exp [\kappa(y + \eta)]\{\exp [i\kappa(x - \xi)] + \exp [-i\kappa(x - \xi)]\} \, d\kappa. \tag{12}$$

Combining equations (12) and (10), we arrive at

$$G = \nu \int_0^\infty \frac{\exp [\kappa(y + \eta)]}{\tau^2\kappa^2 + [2\tau(1 - i\beta) - 1]\nu\kappa + \nu^2(1 - 2i\beta)} \exp [-i\kappa(x - \xi)] \, d\kappa + \nu \int_0^\infty \frac{\exp [\kappa(y + \eta)]}{\tau^2\kappa^2 - [2\tau(1 - i\beta) + 1]\nu\kappa + \nu^2(1 - 2i\beta)} \exp [i\kappa(x - \xi)] \, d\kappa \tag{13}$$

which indeed satisfies both the differential equation (4) and the boundary condition (10), and is complex harmonic, in the form $f(z) + g(\bar{z})$ as expected.

Finally, we put the disturbance velocity potential in the following form:

$$\Phi = \Re\left\{\ln \left| \frac{z - \zeta}{z - \bar{\zeta}} \right| + G(x, y; \xi, \eta)\right\} \exp (i\omega t). \tag{14}$$

2. Wave propagation. From the disturbance velocity potential (14), it is easy to see that the only term that contributes to the propagating wave is $G(x, y; \xi, \eta)$ as given by (13). Now both integrals of G are singular; principal values must therefore be considered.

The integrand of first integral has singularities at the roots of equation

$$\tau^2\kappa^2 - [1 - 2\tau(1 - i\beta)]\nu\kappa + \nu^2(1 - 2i\beta) = 0$$

i.e., at

$$\kappa_{1,2} = [1 - 2\tau(1 - i\beta) \pm (1 - 4\tau + 4\tau i\beta - 4\tau^2\beta^2)^{1/2}]\nu/2\tau^2. \tag{15}$$

When $\beta = 0$, these roots are

$$\kappa_{1,2} = [1 - 2\tau \pm (1 - 4\tau)^{1/2}]\nu/2\tau^2. \tag{16}$$

Similarly, the integrand of second integral has singularities at the roots of equation

$$\tau^2\kappa^2 - [1 + 2\tau(1 - i\beta)]\nu\kappa + \nu^2(1 - 2i\beta) = 0$$

i.e., at

$$\kappa_{3,4} = [1 + 2\tau(1 - i\beta) \pm (1 + 4\tau - 4\tau i\beta - 4\tau^2\beta^2)^{1/2}]\nu/2\tau^2. \tag{17}$$

When $\beta = 0$, these roots are

$$\kappa_{3,4} = [1 + 2\tau \pm (1 + 4\tau)^{1/2}]\nu/2\tau^2. \tag{18}$$

TABLE I

	$\Re$				$\mathcal{I}$			
	κ_1	κ_2	κ_3	κ_4	κ_1	κ_2	κ_3	κ_4
0	g/c^2	0	g/c^2	0	0+	0	0−	0
$\to \frac{1}{4}$	(+)	(+)	(+)	(+)	0+	0+	0−	0−
$\frac{1}{4} \to \frac{1}{2}$	(+)	(+)	(+)	(+)	(+)	(−)	0−	0−
$\frac{1}{2} \to \infty$	(−)	(−)	(+)	(+)	(+)	(−)	0−	0−

A careful investigation of (15) and (17) shows that as long as $\beta \neq 0$, the following results hold as β approaches zero.

In the table, symbol $0\pm$ indicates the following limiting behavior:

$$\lim_{\beta \to 0} \vartheta_{\kappa_i} = 0 \tag{19}$$

As will be shown later, this limit physically corresponds to the case of an undamped propagation of a harmonic wave train of wave length $2\pi/\kappa_i$. Indeed, by writing (13) in the form

$$\begin{aligned} G &= \frac{\nu}{\tau^2}\left[\int_0^\infty \frac{\exp\left[\kappa(y+\eta)\right]}{(\kappa-\kappa_1)(\kappa-\kappa_2)} \exp\left[-i\kappa(x-\xi)\right] d\kappa \right. \\ &\qquad \left. + \int_0^\infty \frac{\exp\left[\kappa(y+\eta)\right]}{(\kappa-\kappa_3)(\kappa-\kappa_4)} \exp\left[i\kappa(x-\xi)\right] d\kappa \right] \\ &= \frac{\nu}{(1-4\tau)^{1/2}} \int_0^\infty \left(\frac{1}{\kappa-\kappa_1} - \frac{1}{\kappa-\kappa_2}\right) \exp\left[\kappa(y+\eta)\right] \exp\left[-i\kappa(x-\xi)\right] d\kappa \\ &\quad + \frac{\nu}{(1+4\tau)^{1/2}} \int_0^\infty \left(\frac{1}{\kappa-\kappa_3} - \frac{1}{\kappa-\kappa_4}\right) \exp\left[\kappa(y+\eta)\right] \exp\left[i\kappa(x-\xi)\right] d\kappa \\ &= \nu\{(1-4\tau)^{-1/2} I_1 + (1+4\tau)^{-1/2} I_2\} \end{aligned} \tag{20}$$

and putting $\Omega = \kappa + im$, all the integrals in (20) can readily be evaluated by proper choice of the contour of integration and application of Cauchy's integral theorem in the complex Ω plane. To fix the proper contour of integration, observe that, in order to make the integral meaningful, in addition to the necessary part of the positive real axis ($m = 0$, $0 < k < \infty$) the remaining part of the contour must consist partly of the imaginary axis ($k = 0$), (as the integral diverges along negative real axis due to the presence of the factor $\exp\left[\kappa(y+\eta)\right]$ under the integral sign), and partly of a circular arc of infinite radius over which the integral vanishes. The choice of the proper quadrant for the contour Γ_i for I_i depends entirely on the sign of the difference $x - \xi$. When $x - \xi$

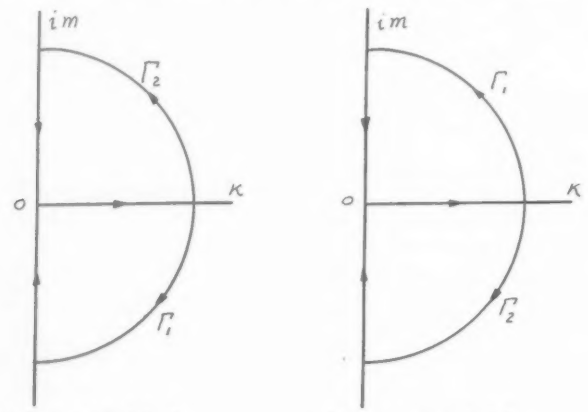

FIG. 1. Contours of integration

is negative, this part of the contour must be taken in the first quadrant of the complex Ω plane for I_1, and in the fourth quadrant for I_2, in order to insure convergence of the integral; and vice versa when $x - \xi$ is positive (see Fig. 1). Whenever a pole k_j ($j = 1, 2$ for I_1, $j = 3, 4$ for I_2) lies inside its corresponding contour of integration Γ_i ($i = 1, 2$), there will be a residue term by Cauchy's integral theorem. As will be shown later, this gives rise to a harmonic wave train which propagates in an undamped or damped manner, according to whether the pole is on or off the real axis, when β approaches zero.

Taking cognisance of the above remarks, and applying Cauchy's integral theorem, we readily write the general expressions for the different cases as follows:

(a) $\underline{x - \xi > 0}$

$$\begin{aligned} I_1 &= \int_0^\infty \frac{\exp[\kappa(y+\eta)]}{\kappa - \kappa_j} \exp[-i\kappa(x-\xi)]\, d\kappa \qquad (j = 1, 2) \\ &= -2\pi i \exp[\kappa_j(y+\eta) - i\kappa_j(x-\xi)] \\ &\qquad + \int_0^\infty \frac{\exp[-im(y+\eta)]}{m - i\kappa_j} \exp[-m(x-\xi)]\, dm, \end{aligned} \tag{21}$$

$$\begin{aligned} I_2 &= \int_0^\infty \frac{\exp[\kappa(y+\eta)]}{\kappa - \kappa_j} \exp[i\kappa(x-\xi)]\, d\kappa \qquad (j = 3, 4) \\ &= 2\pi i \exp[\kappa_j(y+\eta) + i\kappa_j(x-\xi)] \\ &\qquad + \int_0^\infty \frac{\exp[im(y+\eta)]}{m + i\kappa_j} \exp[-m(x-\xi)]\, dm, \end{aligned} \tag{22}$$

(b) $\underline{x - \xi < 0}$

$$\begin{aligned} I_1 &= \int_0^\infty \frac{\exp[\kappa(y+\eta)]}{\kappa - \kappa_j} \exp[-i\kappa(x-\xi)]\, d\kappa \qquad (j = 1, 2) \\ &= 2\pi i \exp[\kappa_j(y+\eta) - i\kappa_j(x-\xi)] \\ &\qquad + \int_0^\infty \frac{\exp[im(y+\eta)]}{m + i\kappa_j} \exp[m(x-\xi)]\, dm, \end{aligned} \tag{23}$$

$$\begin{aligned} I_2 &= \int_0^\infty \frac{\exp[\kappa(y+\eta)]}{\kappa - \kappa_j} \exp[i\kappa(x-\xi)]\, d\kappa \qquad (j = 3, 4) \\ &= -2\pi i \exp[\kappa_j(y+\eta) + i\kappa_j(x-\xi)] \\ &\qquad + \int_0^\infty \frac{\exp[-im(y+\eta)]}{m - i\kappa_j} \exp[m(x-\xi)]\, dm. \end{aligned} \tag{24}$$

It can be shown that all the definite integrals obtained above are regular, and they decrease rapidly with increasing $|x - \xi|$, hence they can only give rise to a local disturbance. Thus it is the residue term only that can lead to infinite propagation of harmonic wave trains. Indeed, noting that all the residues are given in the form

$$2\pi i \exp[\kappa_j(y+\eta) \pm i\kappa_j(x-\xi)],$$

we can readily conclude that an undamped harmonic wave train appears whenever we

have $\lim_{\beta\to 0} \vartheta\kappa_i = 0\pm$. By further taking into account the detailed distribution of singularities in the various frequency ranges as listed in Table I, we obtain the following results:

(a) For $x - \xi > 0$ (i.e. upstream of the travelling source):

I_2: Since both κ_3 , κ_4 lie outside of contour of integration Γ_2 for the entire range of parameter τ, I_2 has no residue.

I_1: i) for $\tau < 1/4$, both κ_1 , κ_2 lie outside of contour Γ_1 , I_1 has no residue.

ii) for $1/4 < \tau < 1/2$, κ_2 lies inside Γ_1 , so that I_1 has a residue term at κ_2 . However, as $\lim_{\beta\to 0} \vartheta\kappa_2 \neq 0$, this residue term leads to a harmonic wave train that is damped out exponentially with $| x - \xi |$. For by writing $\kappa_2 = m - in$, where $n > 0$, we have

$$2\pi i \exp [\kappa_2(y + \eta) - i\kappa_2(x - \xi)]$$
$$= 2\pi i \exp [m(y + \eta) - n(x - \xi)] \exp \{-i[m(x - \xi) + n(y + \eta)]\}$$

which verifies our statement.

iii) for $1/2 < \tau$, both κ_1 , κ_2 lie outside of contour Γ_1 , so that I_1 has no residue.

Thus, on the upstream side of the travelling source, the disturbance is always a local one.

(b) For $x - \xi < 0$ (i.e. downstream of travelling source):

I_2: For the entire range of τ, both κ_3 and κ_4 lie inside the contour of integration Γ_2 , so that I_2 has two residue terms at κ_3 and κ_4 . As $\lim_{\beta\to 0} \vartheta\kappa_3$, $\kappa_4 = 0$, there will be two undamped harmonic wave trains of wave lengths $2\pi/\kappa_3$ and $2\pi/\kappa_4$ on downstream side which propagate to infinity. Indeed, it can easily be shown that

$$I_2 = 2\pi i \{\exp [\kappa_4(y + \eta) + i\kappa_4(x - \xi)] - \exp [\kappa_3(y + \eta) + i\kappa_3(x - \xi)]\}$$
$$+ \int_0^\infty \left(\frac{1}{m - i\kappa_3} - \frac{1}{m - i\kappa_4}\right) \exp [-im(y + \eta) + m(x - \xi)]\, dm$$

which varifies our statement.

I_1: i) for $\tau < 1/4$, both κ_1 and κ_2 lie inside of contour Γ_1 , so that I_1 has two residue terms at κ_1 and κ_2 . As $\lim_{\beta\to 0} \vartheta\kappa_1$, $\kappa_2 = 0$, two undamped harmonic wave trains of wave lengths $2\pi/\kappa_1$, and $2\pi/\kappa_2$ propagate to infinity on the downstream side. This can easily be seen as follows:

$$I_1 = 2\pi i \{\exp [\kappa_1(y + \eta) - i\kappa_1(x - \xi)] - \exp [\kappa_2(y + \eta) - i\kappa_2(x - \xi)]\}$$
$$+ \int_0^\infty \left(\frac{1}{m + i\kappa_1} - \frac{1}{m + i\kappa_2}\right) \exp [im(y + \eta) + m(x - \xi)]\, dm$$

ii) for $1/4 < \tau < 1/2$, only k_1 lies inside the contour Γ_1 , so I_1 has a single residue there. However, as $\lim_{\beta\to 0} \vartheta\kappa_1 \neq 0$, the corresponding harmonic wave train is rapidly damped out.

iii) for $1/2 < \tau$, both κ_1 and κ_2 lie outside of the contour Γ_1 , so that I_1 has no residue.

Thus, on the downstream side of the travelling source, there appear multiple wave trains propagating to infinity. The number of these infinitely propagating wave trains is determined by the parameter τ.

The case of a travelling source of steady strength can easily be checked with Rayleigh's result. Here the poles are at κ_1 and κ_3 (Table I). Now, for points upstream, $x - \xi > 0$, κ_1 lies outside the contour Γ_1 , κ_3 outside Γ_2 , so neither I_1 nor I_2 has any residue. For points downstream, $x - \xi < 0$, however, both κ_1 and κ_3 lie inside their respective contours Γ_1 and Γ_2 , and so both I_1 and I_2 have residues. As $\lim_{\beta \to 0} \vartheta \kappa_1$, $\kappa_3 = 0$, these residue terms represent undamped harmonic wave trains on the downstream side, of wave lengths $2\pi/\kappa_1$ and $2\pi/\kappa_3$ respectively. However, from Table I, we have $\kappa_1 = \kappa_3 = g/c^2$. Thus we only obtain a single harmonic wave train of wave length $2\pi c^2/g$, as expected.

Summarizing, we have the following statements.

When a point source of constant strength travels with speed c beneath the free surface of a two dimensional, ideal incompressible fluid of infinite depth and extent, aside from a local disturbance that travels together with the source, there is also produced a fundamental harmonic wave train of wave length $\lambda_0 = 2\pi c^2/g$, or of frequency $\omega_0 = g/c$, which propagates to infinity on the downstream side. There is no disturbance propagating to infinity on the upstream side. This was first observed by Rayleigh, and agrees exactly with the present analysis ($\tau = 0$).

When the strength of this travelling source fluctuates harmonically, then, aside from the local disturbance that travels with it, there are produced on the downstream side multiple wave trains propagating to infinity. This phenomenon may be regarded as a splitting of the original single wave for constant strength source by the fluctuation of source strength. The number of these infinitely propagating wave trains is determined by the value of a parameter τ, which is the ratio between the fluctuation frequency ω of source strength and the frequency of fundamental wave ω_0 explained above ($\tau = \omega/\omega_0$). When $\tau < 1/4$, there are four wave trains of different wave lengths propagating to infinity on the downstream side; when $\tau > 1/4$, there are only two. No disturbance propagates to infinity on the upstream side.

Inspection of formula (20) reveals an interesting phase of the wave production by a travelling pulsating source. It is seen that the amplitude of both local disturbance and propagated wave trains contributed by the I_1-term exhibits a resonance behavior as the parameter τ approaches its critical value $\tau = 1/4$. Thus violent disturbances should be expected at $\tau = 1/4$ according to the present theory.

The present analysis also agrees with the physical intuition that no disturbance could propagate to infinity on the upstream side. This result can evidently be considered as a justification of our introducing the asymptotic upstream condition of "vanishing disturbance at infinity" as an alternative to Rayleigh's dissipative forces in rendering the solution unique.

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BOOK REVIEWS

Randwertprobleme der Mikrowellenphysik. By Fritz E. Borgnis and Charles H. Papas. Springer-Verlag, Berlin, Göttingen, Heidelberg, 1955. xv + 266 pp. $11.50.

This book is an excellent short treatise on the modern applications of electromagnetic theory to antennas, wave guides, and scattering. The well-organized collection of problems appears for the first time in book form, although much of it could be found by an exhaustive search through the journal literature.

The connecting link between these various problems seems to be the nature of the mathematical techniques needed in their solutions. There is extensive use of Green's functions and their various representations, integral equations, and stationary principles of the "Levine-Schwinger" type. Computational details are often omitted but numerical results are given in graphical form and there is usually a careful discussion of the physical significance of these results.

The very recent and important results on high frequency scattering (work of Keller, Rubinow and Wu, Kodis) apparently could not be included in time for publication. These papers must have reached the authors in proof since there is a footnote throwing doubt on the formula (8.7 on page 51) for the scattering cross section of a circular cylinder. Indeed this result must be discarded and replaced by the one found by Rubinow and Wu. It would also seem that the integral equation (5.11) might be discussed somewhat more carefully since the kernel, which is a second derivative of the Green's function at the singularity, does not exist. The proper limiting procedure in this case has been explained by Bouwkamp in his review article in the "Proceedings of the Royal Society."

Mention should be made of the authors' lucid style; their German is surprisingly easy to follow. The printing is of the highest quality and altogether this volume represents a successful contribution to the literature in the field.

I. Stakgold

Numerical analysis. Edited by John H. Curtiss. Proceedings of Symposia in Applied Mathematics, Vol. 6. McGraw-Hill Book Company, Inc., New York, 1956. vi + 303 pp. $9.75.

The volume contains all but two of the papers presented at the Sixth Symposium in Applied Mathematics of the American Mathematical Society (Santa Monica City College, August 1953). As the space available for this review rules out a discussion of the individual contributions, the diversified contents of the volume is indicated by the following list of papers: Computational Problems in the Theory of Dynamic Programming (R. Bellman), Some Methods for Solutions of Boundary-value Problems of Linear Partial Differential Equations (S. Bergman), Computational Aspects of Certain Combinatorial Problems (R. H. Bruck), Numerical Results on the Shock Configuration in Mach Reflection (D. R. Clutterham and A. H. Taub), Some Applications of Gradient Methods (J. W. Fischbach), Some Qualitative Comments on Stability Considerations in Partial Difference Equations (S. P. Frankel), Approximations in Numerical Analysis—A Report on a Study (C. Hastings, Jr., J. Hayward, and J. P. Wong, Jr.), The Conjugate-gradient Method for Solving Linear Systems (M. R. Hestenes), Number Theory on the SWAC (E. Lehmer), The Assignment Problem (T. S. Motzkin), The Method of Steepest Descent (P. C. Rosenbloom), Function Spaces and Approximation (A. Sard), Some Computational Problems in Algebraic Number Theory (O. Taussky), Machine Attacks on Problems Whose Variables Are Permutations (C. B. Tompkins), Best-approximation Polynomials of Given Degree (J. L. Walsh), Recent Results in Numerical Methods of Conformal Mapping (S. E. Warschawski), On the Asymptotic Transformation of Certain Distributions into the Normal Distribution (W. Wasow), Error Bounds for Eigenvalues of Symmetric Integral Equations (H. Wielandt), On the Solution of Linear Systems by Iteration (D. Young).

W. Prager

(*Continued on p. 282*)

ON NON-EXISTENCE OF RECTILINEAR MOTION IN PLASTIC SOLIDS AND NON-NEWTONIAN FLUIDS*

BY

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1. Introduction. For the class of materials described by Rivlin's theory of non-Newtonian fluids [1], Ericksen [2] has shown in the case of steady rectilinear motion that the speed is overdetermined unless the two scalars which occur in Rivlin's theory satisfy certain definite relations. When the speed is overdetermined, Ericksen conjectures that the only types of steady rectilinear motion possible are either rigid or such that the surfaces of constant speed are planes or portions of right circular cylinders. We show here that this conjecture is correct for the class of Rivlin fluids characterized by the requirement that the stress be unaffected by uniform changes in the speed. This requirement is one which is frequently imposed on theories of plasticity and in this case produces a theory very similar to the theory of perfectly plastic solids.

In addition, we consider the case of steady rectilinear motion through closed cylinders, wherein the speed is not overdetermined. Restricting our attention to the case in which the stress and speed are well behaved, we show that certain conditions on the behavior of the scalars which occur in Rivlin's equations must be satisfied in order that such a motion exist in every closed, smooth cylinder.

In the case of steady non-rectilinear motion through a tube of elliptical cross-section, Green and Rivlin [3] have obtained approximate solutions for a particular class of materials which are included in Rivlin's theory. Here, there is a rectilinear motion parallel to the axis of the tube on which is superimposed a vortex-like motion in each of the four quadrants of the tube's cross-section.

2. Equations of motion. In Cartesian tensor notation, Rivlin's equations are

$$t_{ij} = -p\delta_{ij} + F_1(\mathrm{II}, \mathrm{III})\, d_{ij} + F_2(\mathrm{II}, \mathrm{III})\, d_{ik}\, d_{kj} \tag{1}$$

$$t_{ij,j} + \rho h_i = \rho(\partial v_i/\partial t + v_{i,j}v_j) \tag{2}$$

$$v_{i,i} = 0 \tag{3}$$

where t_{ij} is the stress tensor, d_{ij} is the rate of deformation tensor, defined in terms of the components of velocity v_i by $d_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i})$, $2\mathrm{II} = -d_{ij}d_{ij}$, $\mathrm{III} = \det. d_{ij}$, h_i is the body force per unit mass, p is an arbitrary hydrostatic pressure, F_1 and F_2 are scalar functions of the indicated arguments, and ρ is the density of the material which is assumed to be constant.

The analytic equivalent of the requirement that the stress be unaffected by uniform changes in the speed is that the function $f_{ij}(d_{kl}) = F_1(\mathrm{II}, \mathrm{III})d_{ij} + F_2(\mathrm{II}, \mathrm{III})d_{ik}d_{kl}$ satisfy the relation $f_{ij}(ad_{kl}) = f_{ij}(d_{kl})$ for all $a > 0$. In this event, it can be shown that the scalars $F_1(\mathrm{II}, \mathrm{III})$ and $F_2(\mathrm{II}, \mathrm{III})$ are of the form $G_1(\mathrm{III}/-\mathrm{II}^{3/2})[-2\mathrm{II}^{-1/2}]$ and $G_2(\mathrm{III}/-\mathrm{II}^{3/2})[-2\mathrm{II}]^{-1}$ respectively. Requiring that Eq. (3) hold, we have for the case of rectilinear motion where each particle moves parallel to the x^3 axis of a rectangular coordinate system at constant speed, $v_1 = v_2 = 0$, $v_3 = 2f(x^1, x^2)$; so that

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the quantities II and III are given by $\mathrm{II} = -[f_{,1}^2 + f_{,2}^2]$ and $\mathrm{III} = 0$. Restricting ourselves to the case in which $-II > 0$, we then have from Eqs. (1) and (2) respectively

$$\| t_{ij} \| = -p \| \delta_{ij} \| + \frac{G_1(0)}{(-2\mathrm{II})^{1/2}} \begin{Vmatrix} 0 & 0 & f_{,1} \\ 0 & 0 & f_{,2} \\ f_{,1} & f_{,2} & 0 \end{Vmatrix} + \frac{G_2(0)}{[-2\mathrm{II}]} \begin{Vmatrix} f_{,2}^2 & f_{,1}f_{,2} & 0 \\ f_{,1}f_{,2} & f_{,2}^2 & 0 \\ 0 & 0 & f_{,1}^2 + f_{,2}^2 \end{Vmatrix} \tag{4}$$

and

$$t_{ij,j} + \rho h_i = 0. \tag{5}$$

Assuming that the body forces h_i are conservative so that h_i can be replaced by $\Phi_{,i}$ and carrying out the differentiation called for in Eq. (5), we obtain the following three equations which describe the motion

$$(p - \rho\Phi)_{,1} = G_2(0)\{(f_{,1}^2[-2\mathrm{II}]^{-1})_{,1} + (f_{,1}f_{,2}[-2\mathrm{II}]^{-1})_{,2}\} \tag{6a}$$

$$(p - \rho\Phi)_{,2} = G_2(0)\{(f_{,1}f_{,2}[-2\mathrm{II}]^{-1})_{,1} + (f_{,2}^2[-2\mathrm{II}]^{-1})_{,2}\} \tag{6b}$$

$$(p - \rho\Phi)_{,3} = G_1(0)\{(f_{,1}[-2\mathrm{II}]^{-1/2})_{,1} + (f_{,2}[-2\mathrm{II}]^{-1/2})_{,2}\}. \tag{6c}$$

Since f is a function of x^1 and x^2 only and $G_1(0)$ and $G_2(0)$ are constants, it follows from Eqs. (6) that $p - \rho\Phi$ is of the form

$$Ax^3 + X(x^1, x^2). \tag{7}$$

If we assume that the rate of dissipation of energy, $t_i^j d_i^j$, is non-negative in the region of motion, then it follows from Eq. (4) that $G_1(0)$ is positive. When this is the case, the curvature of the curves of constant speed is constant, since the bracketed expression in the right-hand side of Eq. (6c) is the expression for the curvature of the curves $f(x^1, x^2) = \text{constant}$ [4]. We note here that this result depends only on the form of $F_1(\mathrm{II})$; namely, $K[-\mathrm{II}]^{-1/2}$ where K is a positive constant.

3. The case $G_2(0) = 0$. When $A \neq 0$, we obtain a solution for any function $f(x^1, x^2)$ for which the curves $f(x^1, x^2) = \text{constant}$ are circles of constant radius. For example, the function $f(x^1, x^2) = [x^2 + (r_0^2 - x^{1^2})^{1/2} - r_0]$ gives a solution in the region bounded by the planes $x^1 = -r_0$ and $x^1 = r_0$. Outside of this region the curves $f(x^1, x^2)$ clearly intersect so that the solution is meaningless. It is easily seen in such a motion that the material cannot adhere to the wall of a closed rigid cylinder.

When $A = 0$, the curves $f(x^1, x^2) = \text{constant}$ are straight lines (not necessarily parallel), and any function which has this property will give a solution. The most common example of motion of this type is the case of simple shear wherein the curves of constant speed are parallel straight lines.

4. The case where $G_2(0) \neq 0$. We show here that the curves of constant speed are parallel or concurrent straight lines, where we have assumed, as in the case $G_2(0) = 0$, that $-\mathrm{II} > 0$.

Let us make the coordinate transformation $f^i = f^i(x^1, x^2)$, where the curves $f^1 =$

constant are the curves of constant speed and the curves $f^2 = \text{constant}$ are their orthogonal trajectories. The components of the metric tensor for this system are given by $g^{11} = (\partial f^1/\partial x^1)^2 + (\partial f^1/\partial x^2)^2$, $g^{12} = g^{21} = 0$, $g^{22} = (\partial f^2/\partial x^1)^2 + (\partial f^2/\partial x^2)^2$, $g^{11} = 1/g_{11}$, $g^{22} = 1/g_{22}$, and $g_{12} = g_{21} = 0$. In terms of these quantities, the covariant components of a unit normal to the curves $f^1(x^1, x^2) = \text{constant}$ are given by

$$n_1 = (g^{11})^{-1/2} \quad \text{and} \quad n_2 = 0. \tag{9}$$

The contravariant components are given in terms of the covariant components by the relation $n^i = g^{ij}n_j$, so that

$$n^1 = (g^{11})^{1/2} \quad \text{and} \quad n^2 = 0. \tag{10}$$

In this coordinate system, Eqs. (6a) and (6b) are given by

$$(p - \rho\Phi)_{,1} = \tfrac{1}{4}G_2(0)\ \partial/\partial f^1\ (\log g_{22}) \tag{11a}$$

$$(p - \rho\Phi)_{,2} = \tfrac{1}{4}G_2(0)\ \partial/\partial f^2\ (\log g_{11}). \tag{11b}$$

Integrating these equations and equating the results, we obtain

$$\log g_{11}g_{22} = H_1(f^1) + H_2(f^2) \tag{12}$$

so that the product $g_{11}g_{22}$ is given by

$$g_{11}g_{22} = e^{H_1(f^1)} \cdot e^{H_2(f^2)}. \tag{13}$$

Now, let us make the coordinate transformation $u^1 = u^1(f^1)$, $u^2 = u^2(f^2)$ which is defined by the relations $[df^1/du^1]^2 = e^{-H_1(f^1)}$ and $[df^2/du^2]^2 = e^{-H_2(f^2)}$ so that the components of the metric of the new system are given by

$$a_{11} = g_{11}e^{-H_1(f^1)}, \qquad a_{22} = g_{22}e^{-H_2(f^2)}, \quad \text{and} \quad a_{12} = a_{21} = 0. \tag{14}$$

In any coordinate system, Eq. (6c) is given by

$$n^i_{,i} = A(2)^{1/2}/G_1(0) \tag{15}$$

and in this system, it becomes

$$\tfrac{1}{2}(a^{11})^{1/2}a^{22}\ \partial a_{22}/\partial u^1 = A(2)^{1/2}/G_1(0). \tag{16}$$

Letting $B = A(2)^{1/2}/G_1(0)$ and substituting Eqs. (13) and (14) in Eq. (16), we obtain, since $a^{11} = 1/a_{11}$ and $a^{22} = 1/a_{22}$,

$$(a_{22})^{-1/2}\ \partial a_{22}/\partial u^1 = 2B. \tag{17}$$

Integrating, we obtain

$$a_{22} = [Bu^1 + P(u^2)]^2. \tag{18}$$

Now, u^1 and u^2 are coordinates in the plane, so that we can make use of the requirement that the Riemann tensor vanish. Since, from Eqs. (13) and (14), $a_{11}a_{22} = 1$, this requirement is satisfied, as is easily shown, if and only if the following partial differential equation is satisfied

$$\partial^2 a_{11}/\partial u^{2^2} + \partial^2 a_{22}/\partial u^{1^2} = 0. \tag{19}$$

From Eqs. (18) and (19), we obtain

$$\partial^2 a_{11}/\partial u^{2^2} = -2B^2. \tag{20}$$

Integrating twice, we obtain

$$a_{11} = -B^2u^{2^2} + Q_1(u^1)u^2 + Q_2(u^1). \tag{21}$$

Since $a_{11}a_{22} = 1$ for all values of u^1 and u^2, we may write

$$\partial^3 a_{11}/\partial u^1(\partial u^2)^2 \equiv \partial^3 a^{22}/\partial u^1(\partial u^2)^2. \tag{22}$$

Making use of Eqs. (18) and (19), this relation becomes

$$6B[Bu^1 + P(u^2)]^{-4}\, d^2P(u^2)/du^{2^2} - 24B[Bu^1 + P(u^2)]^{-5}[dP(u^2)/du^2]^2 = 0. \tag{23}$$

Since a_{11} and a_{22} are never zero, we multiply by $[Bu^1 + P(u^2)]^5$, so that Eq. (23) reduces to

$$6B[Bu^1 + P(u^2)]\, d^2P(u^2)/du^{2^2} - 24B[dP(u^2)/du^2]^2 = 0. \tag{24}$$

This relation is clearly satisfied for all u^1 and u^2, if and only if $B = 0$ or $P(u^2)$ is a constant. However, when $P(u^2) =$ constant, it follows easily from Eqs. (18) and (21) and the condition $a_{11}a_{22} = 1$ that $B = 0$. Thus, Eq. (24) is satisfied if and only if $B = 0$.

When $B = 0$, it follows from Eq. (17) that a_{22} is a function of u^2 only; hence since $a_{11}a_{22} = 1$, a_{11} must also be a function of u^2 only. We accordingly make the coordinate transformation defined by the relations $du^1/dv^1 = 1$ and $du^2/dv^2 = (a_{22})^{-1/2}$. The components of the metric tensor in this case are given by

$$b_{11} = [b^{11}]^{-1} = T(u^2), \qquad b_{12} = b^{12} = 0, \quad \text{and} \quad b_{22} = [b^{22}]^{-1} = 1. \tag{25}$$

In the v^1, v^2 coordinate system the condition that the Riemann tensor vanish is satisfied if and only if the following differential equation is satisfied.

$$2d^2b_{11}/dv^{2^2} = 1/b_{11}[db_{11}/dv^2]^2. \tag{26}$$

The two solutions to this differential equation are $b_{11} =$ constant and $b_{11} = [Cv^2 + D]^{2^*}$ Making a coordinate transformation of the form $w^1 = \alpha_1 v^1 + \beta_1$, $w^2 = \alpha_2 v^2 + \beta_2$, we can find α_i and β_i such that the components of the corresponding metric tensor are $\bar{b}_{11} = 1$, $\bar{b}_{22} = 1$, and $\bar{b}_{11} = (w^2)^2$, $\bar{b}_{22} = 1$ respectively for $b_{11} =$ constant and $b_{11} = [Cv^2 + D]^2$. When $\bar{b}_{11} = 1$, the w^1, w^2 coordinate system is Cartesian and when $\bar{b}_{11} = [w^2]^2$, it is polar where the curves $w^1 =$ constant are straight lines in both cases. Since each coordinate transformation made in this section has been of the form $t^1 = t^1(s^1)$, $t^2 = t^2(s^2)$, and the curves $w^1 =$ constant are either parallel or concurrent straight lines, it follows that the curves of constant speed are either parallel or concurrent straight lines.

5. Existence conditions. Ericksen has shown for the case of steady rectilinear motion that the speed is not overdetermined when the two scalars which occur in Rivlin's equations are proportional. For this case we show that there are additional restrictions which must be imposed on these scalars in order that steady rectilinear motion be possible in every closed cylinder, when both the speed and stress are well behaved throughout the region of motion. Specifically, we require in every case that the rate of work $t_i^j d_j^i$ be non-negative; that the stress exist and be finite in the region of motion; that the speed of particles on the cylinder wall be zero; and that the speed be continuously differentiable in the region of motion.

For the case of steady rectilinear motion, parallel to the x^3 axis of a rectangular coordinate system where the speed is not overdetermined, Rivlin's equation for the

stress is

$$\begin{aligned} \| t_{ij} \| = -p \| \delta_{ij} \| + F_1(\mathrm{II}) \begin{Vmatrix} 0 & 0 & f_{,1} \\ 0 & 0 & f_{,2} \\ f_{,1} & f_{,2} & 0 \end{Vmatrix} \\ + cF_1(\mathrm{II}) \begin{Vmatrix} f_{,1}^2 & f_{,1}f_{,2} & 0 \\ f_{,1}f_{,2} & f_{,2}^2 & 0 \\ 0 & 0 & f_{,1}^2 + f_{,2}^2 \end{Vmatrix}. \end{aligned} \tag{27}$$

As an immediate consequence of the assumed behavior of the stress and speed, it follows from Eq. (27) that $F_1(\mathrm{II})$ is finite at all points where $-\mathrm{II} = f_{,2}^2 + f_{,1}^2$ does not vanish

We now consider the behavior of $F_1(\mathrm{II})$ at those points in the region of motion where both partial derivatives of f vanish; namely, when II vanishes. That such points exist is guaranteed by the assumptions that f is continuously differentiable throughout the region of motion and is zero on the cylinder wall and by the theorem which states that any continuous function which is constant on the boundary of its region of definition is either constant or has both a maximum and a minimum in this region. If we exclude the case where f is constant and make use of the requirement that the stress exist and be finite throughout the region of motion, then it follows from Eq. (27) that $\lim_{\mathrm{II}\to 0} F_1(\mathrm{II})f_{,1}$ and $\lim_{\mathrm{II}\to 0} F_1(\mathrm{II})f_{,2}$ exist and are finite. When this is the case, it follows that

$$\{[\lim_{\mathrm{II}\to 0} F_1(\mathrm{II})f_{,1}]^2 + [\lim_{\mathrm{II}\to 0} F_1(\mathrm{II})f_{,2}]^2\}^{1/2} = C \quad \text{(a constant)}. \tag{28}$$

Eq. (28) can be written in the equivalent form

$$\lim_{\mathrm{II}\to 0} (-\mathrm{II})^{1/2} F_1(\mathrm{II}) = C. \tag{29}$$

When C is zero, then $\lim_{\mathrm{II}\to 0} F_1(\mathrm{II})f_{,1} = 0$ for $i = 1, 2$ when $C \neq 0$, then $\lim_{\mathrm{II}\to 0} F_1(\mathrm{II})f_{,i} =$ constant for $i = 1, 2$. Multiplying the expression $F_1(\mathrm{II})f_{,i}$ by $(-\mathrm{II}^{1/2})(-\mathrm{II})^{-1/2}$, we obtain.

$$F_1(\mathrm{II})(-\mathrm{II})^{1/2} \cdot f_{,i}(-\mathrm{II})^{-1/2}. \tag{30}$$

Making use of Eq. (29) and taking the limit as II goes to zero, we obtain

$$\lim_{\mathrm{II}\to 0} \quad f_{,i}(-\mathrm{II})^{-1/2} = \text{constant}. \tag{31}$$

Eq. (31) represents the requirement that the components of the unit normal to the curves of constant speed must exist at points where II vanishes. An indication that this condition can not be satisfied for $C \neq 0$ is given in the case of radially symmetric Poiseuille flow [5]. In polar coordinates, the differential equation determining the speed, f, is

$$df/drF_1([df/dr]^2) = -Dr \tag{32}$$

where D is a constant. By symmetry, and since f is continuously differentiable, df/dr clearly vanishes at $r = 0$. In this case, $-\mathrm{II} = (df/dr)^2$ so that Eq. (32) cannot be satisfied unless $\lim_{\mathrm{II}\to 0} F_1(\mathrm{II})\ (-\mathrm{II})^{1/2} = 0$. When this limit is zero, Eq. (32) can be satisfied. For example, when $F([df/dr]^2)$ is given by $K(df/dv)^{-1/2}$ a solution to Eq.

(32) is

$$f = Ar^3 + B \tag{33}$$

where A and B are constants.

The reader should note here that this result does not justify requiring that the constant in Eq. (29) be zero, since it is assumed that the flow is radially symmetric, and we do not show that this is the only type of Poiseuille flow.

In section 2 of this paper, we showed that few types of steady rectilinear motion were possible when $F_1(\mathrm{II})$ is of the form $K(-\mathrm{II})^{-1/2}$, in which case the constant, C, defined in Eq. (29) is non-zero. Accordingly, if we expect steady rectilinear flow to be possible in every closed smooth cylinder, when the stress and speed behave as assumed in this section, then the scalar $F_1(\mathrm{II})$ must be expressible in the form

$$F_1(\mathrm{II}) = K(-\mathrm{II})^{-1/2} + Y(\mathrm{II}) \tag{34}$$

where K is a non-zero constant and Y is a function for which $\lim_{\mathrm{II}\to 0} (-\mathrm{II})^{1/2}\, Y(\mathrm{II}) = 0$, and is finite at all points where II does not vanish.

6. Conclusion. If there are classes of Rivlin fluids for which steady rectilinear motion is possible in every closed smooth cylinder when the stress and speed satisfy the assumptions listed in the previous section, then these materials must satisfy certain definite conditions. These conditions are: (1) the stress be affected by uniform changes in the speed; and (2) the scalar $F_1(\mathrm{II})$ must be expressible in the form $A(-\mathrm{II})^{-1/2} + Y(\mathrm{II})$ where A is a non-negative constant and $Y(\mathrm{II})$ is a function which is finite when $\mathrm{II} \neq 0$ and for which $\lim_{\mathrm{II}\to 0} (-\mathrm{II})^{1/2}\, Y(\mathrm{II}) = 0$.

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ENERGY DISSIPATION AND LINEAR STABILITY*

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1. Let A be an n by n matrix, A^* its Hermitian adjoint, and A_* the matrix $(A + A^*)/2$. In the real field, this means that $A = A_* + S$, where A_* is (real and) symmetric and S is skew-symmetric. Hence, if $C(x, y)$ denotes the (real) bilinear form belonging to $C(=A, A_*, \cdots)$, then, since $S(x, x)$ vanishes identically, $A(x, x) = A_*(x, x)$ holds for every vector x with real components. The restriction to the real field is made only in order to simplify the notations (and, incidentally, it does not involve any restriction at all, since n can be replaced by $2n$).

If A is a continuous function $A(t)$, where $t_0 \leqq t < \infty$, and if $x(t)$ is any solution vector of

$$x' = A(t)x \tag{1}$$

($x' = dx/dt$) which is distinct from the trivial solution $x(t) \equiv 0$, then, as is well-known, $x(t) \neq 0$ holds for every t. The components of $x(t)$ can be assumed to be real, since $A(t)$ is supposed to be real, hence the real and imaginary parts of a complex solution $x(t)$ of (1) are solution vectors. Let (1) be called stable if $|x(t)| <$ const., as $t \to \infty$, holds for the length of every solution of (1) (and for some constant, which depends on the integration constants determining the solution). On the other hand, let

$$x'' = F(t)x \tag{2}$$

($x'' = d^2x/dt^2$) be called stable if not only $|x(t)| <$ const. but also[1] $|x'(t)| <$ Const. holds for all solutions of (2).

2. For a fixed t, let $\lambda = \lambda(t)$ denote the least, and $\mu = \mu(t)$ the greatest, of the n eigenvalues of the symmetric part[2] $A_* = A_*(t)$ of the coefficient matrix of (1). Then the (unilateral) restriction

$$-\infty \leqq \limsup_{T\to\infty} \int_{t_0}^{T} \mu(t)\, dt < \infty \tag{3}$$

is sufficient in order that (1) be stable. With

$$\lambda = \min_{|y|=1} A(y, y), \qquad \mu = \max_{|y|=1} A(y, y) \tag{4}$$

(where t is fixed in $A = A(t)$ when the min and the max are formed). I found this criterion quite a while ago[3] as a by-product of considerations on non-linear generalizations of (1).

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[1] Concerning various types of linear stability in the scalar case, cf. A. Wintner, Quart. Appl. Math. **13** (1955) 192-195.

[2] The point is that only this, the symmetric, part of the coefficient matrix of (1) enters into the stability criterion (3). For a manifestation of the same situation in a different result, cf. A. Wintner, Amer. Journ. Math. **71** (1949) 362-366.

[3] A. Wintner, Amer. Journ. Math. **68** (1946) 557-559. The result and the method of proof were rediscovered by T. Ważewski, Studia Math. **10** (1948) 48-59, and by others. Cf. R. Conti, Rivista di Mat. Univ. Parma **6** (1955) 3-35, where also further references will be found.

Since the proof is very short, and since certain variants of it will be needed when dealing with (2) below, let it be repeated here: On the one hand, (4) is clear from the identity $A(y, y) = A_*(y, y)$ (which is equivalent to the definition of A_*). On the other hand, scalar multiplication of (1) by $x(t)$ shows that $(|x(t)|^2)'/2$ is identical with $A(x, x)$, where $A = A(t)$, $x = x(t)$. But if the trivial solution $x(t) \not\equiv 0$ is excluded, then division by $|x(t)|^2$ is allowed, and so (4) shows that

$$\lambda(t) \leqq (\log |x(t)|)' \leqq \mu(t). \tag{5}$$

Finally, if (3) is assumed, then it is clear from (5) that $\log |x(t)|$, hence $|x(t)|$, must stay bounded as $t \to \infty$.

The simplest instance of (3) is $\mu(t) \leqq 0$, that is, the assumption that the symmetric part $A_*(t)$ of $A(t)$ is non-positive definite at every t. But more than the stability of (1) can then be asserted.[4] In fact, every $|x(t)|$ must then have a finite limit ($\geqq 0$) as $t \to \infty$, since (5) shows that $\log |x(t)|$ is a non-increasing function if $\mu(t) \leqq 0$ throughout, and so the assertion follows from the existence of the lower bound 0 for the length $|x(t)|$. This deduction is mentioned here because a more elaborate form of it will be needed below.

3. The following considerations will replace (1) by (2). For the sake of concreteness, it will be assumed that the matrix $F = F(t)$ is symmetric (that is, that $F = F_*$ for every t), so that (2) is a (non-conservative, but reversible) linear Lagrangian system, with n degrees of freedom, having the kinetic and potential energies $|x'|^2/2$ and $-F(x, x)$ respectively, and the latter contains t explicitly (there is, of course, no loss of generality in the normalization $|x'|^2/2$ which, if all masses are positive and constant, puts all of them equal to 1). The assumptions will be placed on the (time-dependent) eigenvalues of $F = F_*$. Actually, it will be clear from the proofs that the symmetry restriction $F = F_*$ can be omitted if the assumptions are placed on the symmetric part, $F_*(t)$, of the coefficient matrix of (2).

The energy along a path $x = x(t)$ (which actually must be thought of in phase space, rather than in configuration space) is the scalar $h = h(t)$ defined by

$$2h = |x'|^2 - F(x, x). \tag{6}$$

But $h(t) = \text{const.}$ does not hold (unless $F(t) = \text{Const.}$). There is[5] however a simple explicit rule for the dissipation of the energy which, when subject to appropriate assumptions, can be put to about the same use as the conservation of the energy or its "virial" formulation in the classical results (Dirichlet, Jacobi, Hill, Liapounoff) dealing with stability in the conservative case. In the present case, the explicit form of the law of dissipation is particularly simple[6]; it states that, along any solution path $x = x(t)$ of (2), the energy $h = h(t)$ must vary so as to render the relation

$$dh = D(x, x), \quad \text{where} \quad D = dF \quad \text{and} \quad D = D(t), \tag{7}$$

an identity in t.

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[4] In this regard, cf. A. Wintner, Quart. Appl. Math. **8** (1950) 102-104.

[5] For the general (not necessarily linear) case, cf. A. Wintner, Phil. Mag. **39** (1948) 722-728.

[6] Cf. the reference in the preceding footnote.

If $F(t)$ is just continuous, then (7), as it stands, is meaningless, but it becomes meaningful if it is meant in terms of Stieltjes integrations, with reference to the "measures" $dh(t)$ or $D(t)$. For, on the one hand, a Riemann-Stieltjes integral $\int f(t)dg(t)$ exists whenever $g(t)$ is continuous and $f(t)$ is continuously differentiable (hence, of bounded variation) and, on the other hand, not only $f(t) = x(t)$ but also $f(t) = x'(t)$ is continuously differentiable if $x(t)$ is any solution of (2), if $F(t)$ is just continuous.

If the coefficient matrix F of (2), instead of being just continuous, has a continuous derivative (for the sake of shortness, this will be assumed from now on), then (7) can be written in the form

$$h' = F'(x, x), \quad \text{where} \quad h = h(t), \qquad F' = F'(t) \tag{7 bis}$$

(the existence of a continuous derivative h' is part of the statement). In fact, scalar multiplication of (2) by x' leads from (6) to (7 bis), since $x' \cdot x'' = (\mid x' \mid^2)'/2$.

4. The following fact is a corollary: If the quadratic form belonging to the (symmetric) matrix $F'(t)$ is non-positive [non-negative] definite at a fixed t, then the energy $h(t)$ of every solution vector $x = x(t)$ of (2) is a non-increasing [non-descreasing] function at that t.

In fact, if $\alpha = \alpha(t)$ denotes the least, and $\beta = \beta(t)$ the greatest, eigenvalue of $F'(t)$ at a fixed t, then (7 bis) shows that

$$\alpha(t) \mid x(t) \mid^2 \leqq h'(t) \leqq \beta(t) \mid x(t) \mid^2. \tag{8}$$

But (8) contains the assertion, since the assumption is that $\beta(t) \leqq 0$ [or $\alpha(t) \geqq 0$].

Corresponding to the notation in Section 1, let $\lambda = \lambda(t)$ be the least, and $\mu = \mu(t)$ the greatest, eigenvalue of $F = F(t)$ itself; so that (even if x is not a solution vector)

$$\lambda(t) \mid x \mid^2 \leqq F(x, x) \leqq \mu(t) \mid x \mid^2, \quad \text{where} \quad F = F(t). \tag{9}$$

Since $\mu(t) < 0$ [or $\lambda(t) > 0$] means that $F(t)$ is negative [positive] definite, an adaptation of the proof of criterion (3) for the stability of (1) will now lead to the following criterion (i) for the stability of (2):

(i) *If the coefficient matrix $F(t)$ of* (2) *is strictly negative definite for large t, and if $F'(t)$ is non-positive definite at every fixed large t, then* (2) *is stable.*

It is understood that by the first assumption of (i) is meant the existence of a positive constant ϵ satisfying

$$\max_{\mid y \mid = 1} F(y, y) = \mu = \mu(t) \leqq -\epsilon < 0. \tag{10}$$

On the other hand, the second assumption of (i) means that $\beta(t) \leqq 0$ holds throughout and so, since (8) implies that $h'(t) \leqq 0$, the energy $h(t)$ tends either to a finite limit $h(\infty)$ or to $-\infty$. But (10) prevents the second contingency and, what is more, ensures that $h(\infty) \geqq 0$. In fact, since (6) and (9) imply that $2h \geqq \mid x' \mid^2 - \mu \mid x^2 \mid$, it follows from (10) that

$$2\epsilon^{-1} h(t) \geqq \mid x(t) \mid^2 + \mid x'(t) \mid^2, \tag{11}$$

if, without loss of generality, ϵ is chosen to be less than 1. But (11) shows that $h(t)$ cannot become negative. Hence there exists a finite $h(\infty) \geqq 0$. It follows therefore from (11) that $x(t)$ and $x'(t)$ stay bounded as $t \to \infty$. This proves (i).

5. Suppose, for a moment, that (2) is of a single degree of freedom. Then (2) can be written in the form

$$x'' + \omega^2(t) x = 0 \tag{12}$$

if $-\omega^2 = F \leqq 0$ for all t, as in (i). Suppose that

$$\omega(t) \to \infty \quad \text{as} \quad t \to \infty. \tag{13}$$

Then one would expect that the successive waves of every solution $x(t)$ of (12) (waves the widths of which must tend, as $t \to \infty$, to 0 by virtue of Sturm's comparison theorem) will compel the graph of $x(t)$ to be such as to satisfy

$$x(t) \to 0 \quad \text{as} \quad t \to \infty . \tag{14}$$

But it is known that this turns out to be wrong, since even[7] $\limsup | x(t)| = \infty$ is compatible with (13), and that (14) follows from (13) only under supplementary conditions on the regularity of the growth of $\omega(t)$, the simplest such condition being the monotony of $\omega(t)$,

$$d\omega(t) \geqq 0. \tag{15}$$

The known proofs[8] of the sufficiency of (13) and (15) for (14) depend very much on Sturm's separation theorem and cannot, therefore, be adjusted to an extension to the non-scalar case of (2), the case of more than a single degree of freedom. It will now be shown that the more primitive approach, used above, can readily be modified so as to fill in this gap, as follows:

(ii) *If the greatest of the n eigenvalues of $F(t)$ tends to $-\infty$ as $t \to \infty$, and if $F'(t)$ is non-positive definite at every fixed t, then* (14) *holds for every solution vector of* (2).

Needless to say, the assumptions of (ii) reduce to (13) and (15) in the scalar case ($n = 1$). It is also clear that the assumptions of (ii) imply those of (i). But it was seen in the proof (i) that there exists a finite limit $h(\infty) \geqq 0$.

The first assumption of (ii) is that, if ϵ is an arbitrarily small positive number, and if t is large enough, say $t > t_\epsilon$, then $-\mu(t) < 1/\epsilon$. It follows therefore from (9) and (6) that

$$2h(t) \geqq | x'(t) |^2 + | x(t) |^2/\epsilon \quad \text{if} \quad t_\epsilon < t < \infty .$$

Since $| x'(t)|^2 \geqq 0$, and since there exists a finite $h(\infty) \geqq 0$, this implies that

$$2\epsilon h(\infty) \geqq \limsup_{t \to \infty} | x(t) |^2. \tag{16}$$

But the constant $h(\infty)$ and the expression on the right of (16) are independent of ϵ, and ϵ can be chosen arbitrarily small. Hence the assertion, (14), of (ii) follows from (16)

6. The following variant of (i) will now be proved:

If $F(t)$ is strictly negative definite, and if the n eigenvalues of $F'(t)$ (or, what is the same thing, if both functions

$$\min_{|y|=1} F'(y, y), \qquad \max_{|y|=1} F'(y, y) \tag{17}$$

of t) are absolutely integrable,[8] then (2) is stable.

Note that (iii), in contrast to the unilateral assumption of (i), does not assume that the (internal) dissipating action is *steadily* decreasing. In fact, the absolute integrability

[7]For an explicit example, cf. A. Wintner, Journ. Appl. Phys. 18 (1947) 941-942, where a simple proof for (14) under the supplementary assumption (15) is also given.

The restriction (15) is a particular case of a more inclusive condition, formulated by G. Armellini (1935) and proved by L. Tonelli and G. Sansone (1936), a condition (on the "regularity of the growth" of the coefficient function) which, when added to (13), is sufficient to ensure (14) for all solutions of (12); cf. the detailed presentation in Sansone's *Equazioni differenziali nel campo reale*, 2nd ed. (1949), vol. 2, pp. 52-67, where (on p. 53 and p. 61) further references will be found.

[8]Cf. the preceding footnote.

of both functions (17) of t is compatible with both of them coming arbitrarily close to $-\infty$ *and* ∞ as $t \to \infty$.

Actually, the last criterion can greatly be improved, since the assumption placed on the first of the two functions (17) can be omitted entirely, and the *negative part* of the second of the functions (17) does not matter either (this agrees with (i), where that negative part is arbitrary but the positive part is missing). In other words, all that is needed is

$$\int_{t_0}^{\infty} \gamma(t)\, dt < \infty, \tag{18}$$

where, if $\beta(t)$ has the same meaning as in (8),

$$\gamma = \max(0, \beta) \tag{19}$$

(for a fixed t); so that the situation is as follows:

(i bis) *If $F(t)$ is strictly negative definite, and if $\beta(t)$, the greatest eigenvalue of $F'(t)$, satisfies the unilateral restriction* (18), *where $\gamma(t)$ is defined by* (19), *then* (2) *is stable.*

The proof of this extension, (i bis), of (i) requires a combination of the proof of (i) with an adaptation of a device used in another context[9]. First, it is seen from (19) and (8) that, if a t-value is called of the first or of the second kind according as $h'(t) \leqq 0$ or $h'(t) > 0$, then $|h'(t)| \leqq g(t)|x(t)|^2$ for t-values of the second kind. But since $F(t)$ is assumed to be strictly negative definite, it follows, as in the proof of (i), that (11) holds for every t and for a fixed $\epsilon > 0$. Hence, for t-values of the second kind, $|h'(t)|$ is majorized by a constant multiple of $\gamma(t)h(t)$. But (11) also shows that $h(t)$ is positive for all t if the trivial solution, $x(t) \equiv 0$, of (2) is excluded. Hence, the preceding inequality, when combined with the assumption (18), shows that

$$\int_{\Sigma} |d \log h(t)| < \infty \tag{20}$$

where Σ denotes the sequence of all t-intervals which consist of the t-values of the second kind.

It is clear from (20), where $h(t) > 0$, that $\log h(t)$ tends to a finite limit, hence $h(t)$ to a finite positive limit, if t tends to ∞ so as to be confined to the set, Σ, of the t-values of the second kind. On the other hand, if t tends to ∞ on the set of the t-values of the first kind, then $h'(t) \leqq 0$, hence $h(t)$ must then tend either to a finite limit or to $-\infty$. But the second case is impossible, since $h(t) > 0$, by (11). Accordingly, $h(t)$ stays bounded whether t tends to ∞ on Σ or on the complement of Σ. In view of (11), this proves that $x(t)$ and $x'(t)$ are bounded as $t \to \infty$, as claimed by (i bis).

7. It is seen from (19) that the criterion (18) for the stability of (2) can be thought of as the analogue of the criterion (3) for the stability of (1) (except that, for (2), the strict positive definiteness of $-F(t)$ is assumed in (i bis), in order to complement the strict positive definiteness of the kinetic term $|x'|^2$, of (6), the Hamiltonian of (2)).

The traditional connection between (1) and (2) starts with (2), and, if n is the number of components in (2), it replaces n, x and F by $2n$, (x, x') and

$$A = \begin{bmatrix} 0 & I \\ F & 0 \end{bmatrix} \tag{21}$$

[9]For a slightly weaker version, cf. the reference[5].

[10]A. Wintner, Amer. Journ. Math. **69** (1947) 5-13.

respectively (I denotes the n-rowed unit matrix). But there is also another formal connection between (1) and (2), a connection which starts with (1).

Professor W. Prager was good enough to call my attention to a paper which he published in a not easily accessible periodical[11] and in which, starting with a scalar equation of the form (2), he is led, by differentiation of (3), to the consideration of scalar equations (of third or fourth order in x) which are of second order in x' or x''. An interpretation of this procedure in the phase space of (2) suggests a corresponding procedure for (1), as it stands.

The result, though straighforward enough, is quite instructive. In fact, it turns out that *what leads from* (1) *to* (2) (*for any* n) *is precisely the* (*matrix*) *equation of Riccati*:

$$F(t) = A'(t) + A^2(t). \tag{22}$$

In fact, if $A(t)$ is differentiable, then (1) implies that $x'' = A'x + Ax'$, and so (2) follows from (1) if F is defined by (22).

Needless to say, the case (22) of (2) is only a necessary condition for a solution $x(t)$ of (1), simply because the general solution of (1) depends on n, whereas that of (2) depends on $2n$, integration constants. The pitfalls of this situation can be illustrated as follows:

Let $F(t)$ be any n by n matrix which is continuous for large positive t and has the property that, when t is fixed, its symmetric part

$$F_*(t) \text{ is non-negative definite} \tag{23}$$

(for instance, that $F(t)$ is symmetric and non-negative). Then it is known[12] that, if $x(t)$ is any solution vector of (2), the graph of $s = |\, x(t)|^2$ in the (t,s)-plane always turns its convexity toward the t-axis and that, if $x_1(t), \cdots, x_n(t), x_{n+1}(t), \cdots, x_{2n}(t)$ is a complete set of linearly independent solutions of (2), then it is possible to choose $x_1(t), \cdots, x_n(t)$ to be bounded vectors (hence, by convexity, such that $|\, x_1(t)|^2, \cdots, |\, x_n(t)|^2$ tend to finite, non-negative limits as $t \to \infty$), whereas $x_{n+1}(t), \cdots, x_{2n}(t)$ are unbounded (which implies that $|\, x(t)| \to \infty$ holds for any solution vector which is linearly independent of $x_1, \cdots, x_n$).

Let this be applied to the particular case (23). It then follows that, if (22) satisfies (23), the case (22) of (2) has exactly n, linearly independent, solution vectors which are bounded as $t \to \infty$. But none of these n solutions will in general be a solution of (1) also. In fact, if $A(t) = I$ for all t, then (22) satisfies (23), but (1) reduces to $x' = x$ and has, therefore no bounded solution distinct from $x(t) \equiv 0$.

[11] W. Prager, Rev. de la Fac. des Sciences de l'Univ. d'Istanbul, 1 (1936) 37-43.

[12] See the paper referred to under [3].

BOUNDS ON MINIMUM WEIGHT DESIGN*

BY

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Summary. A somewhat limited design procedure for elastic-perfectly plastic structures was developed previously [1]**. It is extended here to provide upper and lower bounds on the minimum weight of three dimensional structures and is specialized to safe one and two dimensional structures in which either direct stresses or bending stresses are negligible. The generalization also includes the influence of body forces. In principle, therefore, such troublesome factors as the weight of the structure itself or centrifugal "forces" may be designed for in a direct manner. Radially symmetric plane stress and plate bending examples are solved to demonstrate direct design procedures.

Introduction. Although design rather than analysis is the real problem in machines and structures, far more research effort is spent on analysis. The reason is primarily the specific nature of the problem posed and the greater possibility of obtaining an unambiguous solution. In what follows the material of construction is idealized as elastic-perfectly plastic. This first approximation to the behavior of real structural metals beyond the elastic range and the accompanying techniques of limit analysis are suitable for problems in which load carrying capacity is of primary importance. In addition to providing more realistic answers for ductile materials, the great advantage of limit analysis over elastic analysis is its relative simplicity. The assumption of perfect plasticity opens the possibility of direct design in the strict sense as opposed to preliminary guess and repeated analysis with the end point not necessarily the best that can be achieved. For problems of such difficulty that direct plastic design is not feasible, it is possible to obtain bounds on an optimum design. Such bounds provide a sound basis of comparison with any proposed structure and indicate the gains, if any, which can be achieved by further refinement.

An economical structure ordinarily will be one of least weight within the restrictions of the procedures of fabrication. Framed steel structures, for example, often are best constructed of beams of constant cross-section between joints. Heyman [2], Foulkes [3], Prager [4] and Livesley [5] have studied minimum weight frames of this type. The present paper dealing with beams, plates, sheets, and space structures is in the spirit of Michell [6] where the minimum weight is sought without regard to problems and costs of manufacture and construction. Prior studies of plates in bending from this point of view have been made by Hopkins and Prager [7] [8], Freiberger and Tekinalp [9].

A start on a broad theory was made [1] by the establishment of a criterion for absolute minimum weight design for structures which are subjected to direct or membrane stresses and for relative minimum weight in the case of beams or plates in transverse bending. These criteria which include that of Michell as a special case suffer from the same disadvantage. They cannot be satisfied in all, and probably not in most, problems which arise. As part of the extension of this earlier work, the matter of existence of

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**Numbers in square brackets refer to the bibliography at the end of the paper.

solutions will be explored briefly, but major emphasis will be on the inclusion of body force and the obtaining of bounds.

2. The problem and its relation to limit analysis. The theorems of limit analysis [10] are concerned with whether a body of elastic-perfectly plastic material will collapse or not under given loads. The term collapse is used here to describe conditions for which plastic flow would occur under constant loads if the accompanying changes in the geometry of the body were disregarded. The boundary conditions are assumed to be of the stress type. That is, each of the components T_i of the surface traction is prescribed except where the corresponding velocity component is prescribed to be zero. The behavior of the elastic-perfectly plastic material is characterized by a positive yield function f which is a function of the stress components σ_{ij} $(i, j = 1, 2, 3)$ referred to rectangular Cartesian coordinates x_i . The material behaves elastically under states of stress for which $f < k^2$, where k is a constant. Plastic flow can occur under states of stress for which $f = k^2$. States of stress for which $f > k^2$ are not permissible.

The yield condition $f(\sigma_{ij}) = k^2$ can be considered to define a surface in stress space in which the components of stress are used as coordinates. Plastic flow can occur under a state of stress represented by a point on this surface. In the same space, a plastic strain *rate* ϵ^p_{ij} can be represented by a vector with components proportional to the components of the plastic strain rate. For a perfectly plastic material the vector representing the plastic strain rate ϵ^p_{ij} associated with a plastic state of stress σ_{ij} is in the direction of the exterior normal to the yield surface at the stress point σ_{ij} . If the yield surface has a singular point where there is not a unique normal, then the strain rate vector must lie in the fan bounded by the normals to the yield surface at adjacent points [11].

The rate D of dissipation of energy per unit volume of the material due to plastic action is given by $\sigma_{ij}\epsilon^p_{ij}$. For a given yield function f, the plastic flow rule enables the dissipation rate D to be expressed as a function of the plastic strain rate only,

$$D = \sigma_{ij}\epsilon^p_{ij} = D(\epsilon^p_{ij}). \tag{1}$$

The term safe statically admissible stress field will be used to denote a stress field σ_{ij} which satisfies the equations of equilibrium

$$\sigma_{ij,j} + F_i = 0 \tag{2}$$

throughout the volume V of the body, satisfies the boundary conditions

$$\sigma_{ij}n_j = T_i \tag{3}$$

on the surface S where the tractions T_i are prescribed, and which is below yield everywhere in V, $f(\sigma_{ij}) < k^2$. In (2), F_i are the components of the body force per unit volume of the material and in (3), n_j are the components of the unit normal to the surface S. Repeated subscripts in the same term denote summation.

A *velocity* field u_i will be called kinematically admissible if it represents a possible plastic flow in the body and if it vanishes on those portions of the surface of the body where the surface traction is not prescribed.

The following theorems have been formulated [10] for an elastic-perfectly plastic body under surface tractions T_i .

Theorem 1. *If a safe statically admissible stress field* σ^S_{ij} *can be found then the body will not collapse under the loads* T_i .

Theorem 2. Collapse must impend or have taken place previously if a kinematically admissible velocity field u_i^k can be found such that the rate at which work is done by the external forces equals or exceeds the dissipation rate, that is for u_i^k continuous in V

$$\int_V D(\epsilon_{ij}^k)\, dV \leq \int_S T_i u_i^k\, dS + \int_V F_i u_i^k\, dV. \tag{4}$$

In (4), the rate of dissipation D is calculated from the strain rates

$$\epsilon_{ij}^k = \tfrac{1}{2}(u_{i,j}^k + u_{j,i}^k), \tag{5}$$

which are treated as purely plastic.

As the basic problem of design for minimum weight we consider a volume V_T within which a structure loaded by a given force distribution is to be placed. The loads T_i are to be supported by some distribution of a given elastic-perfectly plastic material in the volume V_T . The problem is to determine a structure or distribution of the material which is just at the point of collapse under the loads T_i and which involves minimum volume of the material. We shall assume that the material is homogeneous so that the problem of design for minimum volume coincides with the problem of design for minimum weight.

It is apparent that for the problem to admit a solution it must be possible to design at least one structure in V_T capable of supporting the loads T_i . Also the loads T_i will be reached by some loading program and it is presumed that the minimum volume structure would not collapse at any previous stage of the loading program.

We remark that in design for minimum weight, modifications to the approach below can be made to allow for non-homogeneity if it is prescribed that different materials are to be used in different specified regions of the volume V_T .

3. Bounds on the minimum volume. A design based on any safe statically admissible stress field σ_{ij}^S in V_T for the given material will support the loads T_i . The body formed by filling the volume V_S where σ_{ij}^S is non-zero with the given material will not collapse under the loads T_i , by Theorem 1 above. In general the stresses σ_{ij}^S are not the actual stresses in the body V_S subject to the loads T_i . By definition, the magnitude of the volume V_m of a design with minimum volume is bounded from above by the magnitude of V_S , $V_m < V_S$. In practice it is convenient to use stress fields for which $f(\sigma_{ij}^*) \leq k^2$. For such fields $V_m \leq V_*$.

In order to bound V_m from below the second theorem of limit analysis is applied. Since the minimal design is just at collapse under the loads T_i , then by Theorem 2 for any kinematically admissible continuous velocity field u_i^k defined in V_T ,

$$\int_{V_m} D(\epsilon_{ij}^k)\, dV \geq \int_A T_i u_i^k\, dS + \int_{V_m} F_i u_i^k\, dV, \tag{6}$$

where A is the loaded surface. The body forces F_i are assumed to act only if material is present. The equality sign holds in (6) only if u_i^k is a possible collapse mode of the minimum volume design. Inequality (6) can be written

$$\int_{V_m} \Delta(u_i^k)\, dV \geq \int_A T_i u_i^k\, dS, \tag{7}$$

where

$$\Delta(u_i^k) = D(\epsilon_{ij}^k) - F_i u_i^k . \tag{8}$$

A lower bound on V_m follows from (7):

$$V_m \geq V_k = \int_A T_i u_i^k \, dS / \max \Delta(u_i^k), \tag{9}$$

where max Δ (u_i^k) is the maximum value of Δ in V_T which includes V_m . We note that Δ must remain finite in V_T if (9) is to provide a non-zero bound. For a material obeying either the Tresca or von Mises yield condition this requires the velocity components u_i^k to be continuous throughout V_T and to satisfy the condition of incompressibility. For some materials discontinuous velocity fields will be permissible if the material is such that the work done per unit area of the discontinuity surface is zero. Soil which is unable to take tension provides such an example when the discontinuity in velocity is associated with the formation of a tension crack [12].

The lower bounds V_k determined from kinematically admissible velocity fields u_i^k by (9) may prove to be too low to be of value in bracketing V_m . The less restrictive statement (7) may then provide useful information by employing the following procedure. For a given kinematically admissible velocity field u_i^k in V_T , the value of the modified dissipation function Δ (u_i^k) will in general vary throughout V_T . Material is placed in the regions of V_T where Δ takes its largest values until a stage is reached at which material occupies a region V_1 of V_T and

$$\int_{V_1} \Delta(u_i^k) \, dV = \int_A T_i u_i^k \, dS. \tag{10}$$

The surface bounding V_1 will be a surface on which Δ has a constant value. The values of Δ at points in the volume V_1 will not be less than the values Δ assumes at points in V_T not in V_1 . It follows that the volume V_m satisfying the inequality (7) cannot have less volume than V_1 , $V_m \geq V_1$. Equality between the lower bound V_1 and the lower bound V_k given by (9) can only occur if Δ (u_i^k) is constant in the region V_1 so that in general the bound V_1 is an improvement on the bound V_k .

4. Plane stress. The case of plane stress follows the general approach of the previous section. It is required to design a plane sheet of variable thickness which is just at the point of collapse under loads acting in the plane of the sheet and which has minimum volume. The sheet is formed by distributing a given material at points on or at a small distance from a plane middle surface A_T . The variable thickness of the sheet at any point of A_T is denoted by h. Forces $T_i h$ which are independent of h act in the plane of A_T on specified lines L in the plane A_T . The body forces F_i per unit *volume* of the material also act in the plane of A_T .

The middle plane A_T is taken to lie in the plane $x_3 = 0$ for definiteness. The assumption of plane stress requires the stress components σ_{i3} to vanish. The remaining components $\sigma_{\alpha\beta}$ $(\alpha, \beta = 1, 2)$ are functions of x_1 and x_2 only and for equilibrium satisfy the equations

$$(\sigma_{\alpha\beta} h)_{,\beta} + F_\alpha h = 0. \tag{11}$$

The stress boundary conditions on the lines L where the forces $T_\alpha h$ are prescribed require

$$(\sigma_{\alpha\beta} h) n_\beta = T_\alpha h, \tag{12}$$

where n_β are the components of the unit normal to L in the plane $x_3 = 0$.

A kinematically admissible *velocity* field u_i representing a possible plastic flow of a sheet with middle surface A_T is such that u_1 and u_2 are functions of x_1 and x_2 only. The rate D of dissipation of energy per unit *volume* can be found from the velocity components u_1 , u_2 since the stresses σ_{i3} are zero,

$$D = \sigma_{\alpha\beta}\epsilon_{\alpha\beta} = D(\epsilon_{\alpha\beta}). \tag{13}$$

As in Sec. 3, a design based on any statically admissible stress field $\sigma^s_{\alpha\beta}$ will provide an upper bound for the minimum volume V_m . The stress field $\sigma^s_{\alpha\beta}$ must satisfy Eqs. (11) and (12) for some distribution of thickness h_s and must be at or below yield, $f(\sigma^s_{\alpha\beta}) \leq k^2$. The volume V_s ,

$$V_s = \int_{A_T} h_s \, dA, \tag{14}$$

is an upper bound on V_m .

For any kinematically admissible velocity field defined by the components u_1^k , u_2^k over A_T it follows from Theorem 2 that since the minimal design of thickness h_m is just at the point of collapse under the loads

$$\int_{A_T} \Delta(u_\alpha^k) h_m \, dA \geq \int_L (T_\alpha h) u_\alpha^k \, dL, \tag{15}$$

where

$$\Delta(u_\alpha^k) = D(\epsilon_{\alpha\beta}^k) - F_\alpha u_\alpha^k . \tag{16}$$

The lower bound V_k on V_m follows from (15) as before,

$$V_m \geq V_k = \int_L (T_\alpha h) u_\alpha^k \, dL / \max \Delta(u_\alpha^k), \tag{17}$$

where max Δ (u_α^k) is the maximum value of Δ in the plane area A_T .

As described in Sec. 3, the inequality (7) can be used to improve the lower bound (9) and similar remarks apply here with respect to inequality (15) and the lower bound (17). However in this case an upper limit may have to be placed upon the thickness h in order to avoid the somewhat unrealistic line flanges or similar reinforcements of finite cross-sectional area and therefore infinite height.

5. Membranes. In analogous fashion to the case of plane stress, the theory of Sec. 3 can be applied to membrane design. The membrane has a prescribed curved middle surface A_T and supports prescribed loads which in this case may act normal to the surface A_T as well as in the tangent planes to A_T .

6. Special stress and velocity fields. A special case arises if it is possible to design a body V_c in V_T which is compatible with a collapse mode u_i^c such that Δ (u_i^c) is constant throughout V_T which includes V_c . In the absence of body force this is the problem treated in the previous paper [1]. As u_i^c is the actual collapse mode for V_c under the loads T_i ,

$$\int_{V_c} \Delta(u_i^c) \, dV = \int_A T_i u_i^c \, dS. \tag{18}$$

For any other design V_s in V_T just at the point of collapse or capable of carrying the loads T_i , by Theorem 2 above,

$$\int_{V_s} \Delta(u_i^c)\, dV \geq \int_A T_i u_i^c\, dS. \tag{19}$$

As $\Delta\ (u_i^c)$ is constant throughout V_T, comparison of (18) and (19) shows that $V_c \leq V_s$. It follows that a design compatible with such a collapse state is a minimal design $V_c = V_m$. The term compatible means that there exists an equilibrium state of stress which is consistent with the plastic strain rates and which satisfies the boundary conditions on the surface traction and satisfies the yield condition everywhere in V_c.

A similar result applies to the plane stress and membrane cases. For plane stress, a sheet design V_c of thickness h_c compatible with a collapse state for which Δ given by (16) is constant over the middle surface A_T is a design with minimum volume. An example which uses this result is given later in Sec. 8.

In most cases it will not be possible to design a body compatible with a collapse state Δ = constant. This may be seen by analogy with small strain linear and non-linear elasticity. For a positive yield function $f(\sigma_{ij})$ which is homogeneous of degree n in the stress components σ_{ij} and which defines a smooth yield surface, the plastic strain rates are given by

$$\epsilon_{ij}^P = \lambda \frac{\partial f}{\partial \sigma_{ij}}, \tag{20}$$

where λ is a non-negative scalar function of position. In the absence of body force, the dissipation rate $\Delta\ (u_i)$ or $D(\epsilon_{ij}^P)$ is given by

$$D(\epsilon_{ij}^P) = \sigma_{ij}\epsilon_{ij}^P = n\lambda f. \tag{21}$$

The body is at yield everywhere, $f(\sigma_{ij}) = k^2$. It follows that for D to be constant, λ must be constant throughout V_c. For $n = 2$, relation (20) is then formally equivalent to a linear stress-strain relation between elastic strains ϵ_{ij}^P and stresses σ_{ij}, the function λf playing the role of strain-energy density. In general a solution of the equations of elasticity will not have constant strain-energy density, and therefore in general a collapse state with D constant will not exist for the body V_c. For $n > 2$ the analogy is with non-linear elasticity and the function λf is equivalent to the complement of the strain-energy density.

The result $V_c \leq V_s$ also follows from (18) and (19) if $\Delta\ (u_i^c)$ is constant throughout V_c and smaller elsewhere in V_T. A stronger result is the following.

A design occupying the volume V_c compatible with a collapse mode such that the value of Δ at any point in V_c is not less than the value of Δ elsewhere in V_T is a minimum volume design.

A velocity field which is discontinuous across a surface may be considered as the limiting case of a continuous velocity field. The approach of Sec. 3 and the result of the previous paragraph therefore apply to discontinuous velocity fields. The term involving the dissipation of energy at the discontinuity surface can be taken into account explicitly as in Ref. 9. Discontinuous velocity fields may prove useful but only if the total rate of energy dissipation at the discontinuity surface is finite.

When the formal approach of this section is of no practical value, the more general approach of Sec. 3 must be used to obtain either the minimum design or bounds on its volume.

7. Bending of plates. A thin homogeneous plate in bending considered as a three dimensional body provides an example in which Δ cannot be constant throughout.

Here the results of the previous paper [1] are generalized to include body force. In bending, the strain rates $\epsilon_{\alpha\beta}$ in the plate vary linearly throughout the thickness and the rate of dissipation per unit volume is given by

$$D(\epsilon_{\alpha\beta}) = \sigma_{\alpha\beta}\epsilon_{\alpha\beta} = \frac{2 \mid x_3 \mid}{h} D_0 , \tag{22}$$

where D_0 is the value of D at the plate surfaces $x_3 = \pm h/2$. The modified dissipation rate Δ defined by (8) is therefore given by

$$\Delta(w) = \frac{2 \mid x_3 \mid}{h} D_0 - F_3 w, \tag{23}$$

where w is the rate of (normal) deflection of the plate. For a plate, with middle surface A_c and thickness h_c, which is on the point of collapse in a deflection rate pattern w^c

$$\int_A pw^c \, dA + \int_{A_c} F_3 w^c h_c \, dA = \int_{V_c} D(\epsilon^c_{\alpha\beta}) \, dV = \int_{A_c} \tfrac{1}{2} D_0^c h_c \, dA, \tag{24}$$

where p is the transverse load per unit area. For any other plate, with middle surface A_s and thickness h_s, just at collapse or capable of carrying the load p,

$$\int_A pw^c \, dA + \int_{A_s} F_3 w^c h_s \, dA \leq \int_{A_s} \tfrac{1}{2} D_0^s h_s \, dA. \tag{25}$$

From (22),

$$D_0^s = D_0^c \frac{h_s}{h_c} , \tag{26}$$

and it follows from (24) and (25) that

$$\int_{A_c} (\tfrac{1}{2} D_0^c - F_3 w^c) h_c \, dA \leq \int_{A_s} (\tfrac{1}{2} D_0^c \frac{h_s}{h_c} - F_3 w^c) h_s \, dA. \tag{27}$$

We now suppose that the plate with middle surface A_s is any neighboring plate to the plate A_c in the sense that $h_s = h_c + \delta h$ where $\delta h \ll h_c$. The middle surfaces A_s and A_c are then the same, $A_c = A_s$, and if the second order term in (27) is ignored

$$\int_{A_s} (D_0^c - F_3 w^c) \delta h \, dA \geq 0. \tag{28}$$

It follows that if $D_0^c - F_3 w^c$ is constant (and positive) over A_s,

$$\int_{A_c} h_c \, dA \leq \int_{A_s} h_s \, dA. \tag{29}$$

Thus a design compatible with a deflection rate pattern for which Δ is constant on the surfaces of the plate is a *relative* minimum in the solution of the minimum volume problem.

It is more usual to consider a plate in two dimensions rather than three. The corresponding result in terms of bending moments $M_{\alpha\beta}$[9] and rates of curvature $k_{\alpha\beta}$ can be obtained directly or it can be derived from the work above. The rate of dissipation per unit area of the middle surface is given by

$$M_{\alpha\beta} k_{\alpha\beta} = \int_{-h/2}^{h/2} D \, dx_3 = \tfrac{1}{2} D_0 h. \tag{30}$$

A relative minimum is provided by a design compatible with a deflection rate pattern such that $2M_{\alpha\beta}k_{\alpha\beta}/h - F_3 w$ is constant over the plate.

The results for the non-homogeneous sandwich plate are more definite. By a sandwich plate is meant here a plate which has a light-weight core of constant or variable thickness H between thin identical face sheets each of thickness t which varies from point to point of the middle surface A_T. The core carries no bending stresses and a bending moment across the plate is carried by equal and opposite membrane stresses in the outer faces. The problem is to determine the variable thickness t of the faces so that the plate is just at the point of collapse under the prescribed loads and the face sheets have minimum volume. Since H is considered to be large compared with t, $H \gg t$, the strain rates in the faces of the plate can be considered to be constant throughout the thickness t. It follows from the general result of Sec. 6 that a sandwich plate designed to collapse in a mode w such that Δ is constant in the faces of the plate is an absolute minimum volume design.

8. Plane stress example-rotating disc. In the previous paper [1] the formal approach of Sec. 6 was used to obtain minimum volume designs for a circular annular disc loaded by uniform forces per unit length on the inner and outer edges of the disc. Here an illustration is given of the way in which centrifugal "forces" can be taken into account.

A circular annular disc of inner radius r_i and outer radius r_0 is rotating with constant angular velocity ω about a perpendicular axis through the center of the disc. The outer edge of the disc is loaded by a uniform tensile force T per unit length and the inner edge is stress free. The problem is equivalent to that of a stationary disc with outward radial body force $\rho\omega^2 r$ per unit volume, where ρ is the mass density of the material. The material of the disc is assumed to obey Tresca's yield condition, Fig. 1, with yield stress σ_0.

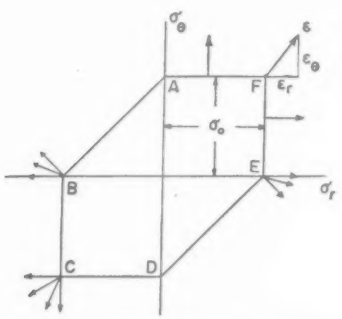

FIG. 1 Tresca's yield condition for plane stress with radial symmetry.

For equilibrium the principal stresses σ_r and σ_θ satisfy

$$\frac{d}{dr}(h\sigma_r) + h\frac{(\sigma_r - \sigma_\theta)}{r} = -\rho\omega^2 rh, \tag{31}$$

where h is the thickness of the disc. In terms of the radial velocity u the strain rates are given by

$$\epsilon_r = du/dr, \qquad \epsilon_\theta = u/r, \tag{32}$$

and for Δ constant,

$$\Delta = \sigma_r \epsilon_r + \sigma_\theta \epsilon_\theta - \rho\omega^2 ru = \text{constant}. \tag{33}$$

The condition Δ = constant restricts the position of the stress point on the yield hexagon of Fig. 1. For example, the stress point cannot lie on the side AF because the normality condition requires $\epsilon_r = 0$, and the condition (33) cannot be satisfied in view of (32). It is readily found that only the stress states represented by points C and F, B and E, can be associated with a velocity field for which Δ is constant for a finite range of r.

In the present problem the stresses are tensile so that the stress points B and C are prohibited. The stress point E is also inadmissible as it would require inward radial velocity at the inner edge $r = r_i$. The remaining stress point F provides the solution. At the stress point F, $\sigma_r = \sigma_\theta = \sigma_0$ and equilibrium requires

$$\frac{dh}{dr} = -\frac{\rho\omega^2}{\sigma_0} rh. \tag{34}$$

The region $r_i < r \leq r_0$ of the disc is at the stress state F so that

$$h = \frac{T}{\sigma_0} \exp(x_0^2 - x^2) \qquad x_i < x \leq x_0 , \tag{35}$$

where $x = (\rho\omega^2/2\sigma_0)^{1/2} r$. In order to complete the design, a line flange must be added at the inner boundary, Fig. 2a, to satisfy the condition of zero stress at $r = r_i$. The line flange is of infinite height but of finite cross-sectional area of amount

$$r_i h_i/(1 - 2x_i^2), \tag{36}$$

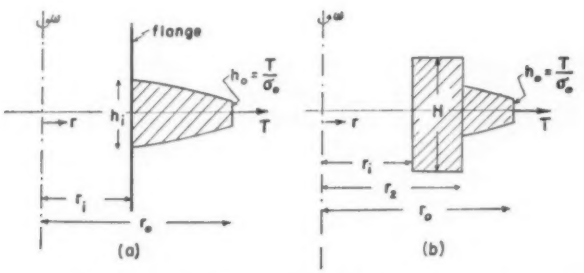

FIG. 2 Minimum volume design for rotating annular disc under exterior tension (a) of unlimited thickness and (b) of restricted thickness.

where h_i is the value of h given by (35) at $x = x_i$. The speed of rotation of the disc is restricted by the condition $x_i^2 < 1/2$, that is $\rho\omega^2 r_i^2 < \sigma_0$.

It remains to associate a velocity field of constant Δ with the design. For the stress point F,

$$\sigma_0 \left(\frac{du}{dr} + \frac{u}{r}\right) - \rho\omega^2 ru = \text{constant} \tag{37}$$

and

$$\epsilon_r = du/dr \geq 0, \qquad \epsilon_\theta = u/r \geq 0. \tag{38}$$

The line flange is at the stress point A so that $du/dr \leq 0$ at $r = r_i$. Taking $du/dr = 0$ at $r = r_i$ gives with Eq. (37)

$$u = \frac{A}{x}\left[\exp\left(x^2 - x_i^2\right)/(1 - 2x_i^2) - 1\right], \tag{39}$$

where A is a positive constant. The restrictions (38) are satisfied for $r \geq r_i$ provided that $x_i^2 < 1/2$.

The volume V_m of the minimum volume design is given by

$$\frac{V_m \rho \omega^2}{2\pi T} = \exp\left(x_0^2 - x_i^2\right)/(1 - 2x_i^2) - 1, \tag{40}$$

and is an increasing function of x_i.

The somewhat unrealistic line flange can be avoided by placing an upper limit H on the thickness h, Fig. 2b. The full thickness H is taken for $r_i \leq r < r_2$ and the stress state in this region is represented by points on the side AF of the yield hexagon, $\sigma_\theta = \sigma_0$, $0 \leq \sigma_r < \sigma_0$. The edge $r = r_2$ of the full thickness region is found from the condition that (σ,h) is continuous at $r = r_2$. The collapse mode must be such that $du/dr = 0$ for $r_i \leq r < r_2$, so that u is constant in this region, $u = B$, and

$$\Delta = B(\sigma_0/r - \rho\omega^2 r) \qquad r_i \leq r < r_2 . \tag{41}$$

Thus Δ is a decreasing function of r for $r_i \leq r < r_2$. For $r_2 < r \leq r_0$ the radial velocity is determined by (37), (38) and the condition $du/dr = 0$ at $r = r_2$,

$$u = \frac{C}{x}\left[\exp\left(x^2 - x_2^2\right)/(1 - 2x_2^2) - 1\right] \qquad x \geq x_2 . \tag{42}$$

The positive constant C is chosen so that u is continuous at $r = r_2$. The value of Δ is constant for $r \geq r_2$ and equal to the value given by (41) at $r = r_2$. A lower limit must be placed on H and an upper limit on the speed of rotation ω for the design to be possible. The design is clearly the minimum volume design but this can also be seen from the following. The design is compatible with a collapse mode for which the value of Δ in the body is not less than the value of Δ elsewhere in the region where material may be placed and is therefore a minimum volume design.

9. Sandwich plate example. The problem of this section illustrates the direct design approach of Sec. 6. A simply supported circular sandwich plate of constant core thickness H is loaded by a uniform pressure p over its upper surface, which is horizontal. It is required to find the variable thickness t of the two identical face sheets so that the plate is just at the point of collapse under the pressure p and the face sheets have minimum volume. The weight per unit volume of the face sheets is denoted by γ and the body force is taken into account in the analysis. We first outline the design procedure for a material with a general yield function and then consider the particular case of the Tresca yield condition.

We write the yield condition on the radial bending moment M and the circumferential bending moment N in the form

$$F(M, N) = M_0^2 , \tag{43}$$

where M_0 is given by

$$M_0 = \sigma_0 H t \tag{44}$$

and σ_0 is the yield stress of the material in tension or compression. The yield function F is homogeneous of order two in the moments M and N and will be assumed to define a smooth curve in the M, N plane. The von Mises yield criterion for example provides the special form $F = M^2 - MN + N^2 = M_0^2$. For equilibrium the moments satisfy the equation

$$\frac{d^2}{dr^2}(rM) - \frac{dN}{dr} + pr + 2\gamma tr = 0, \tag{45}$$

where r measures distance from the center of the plate. Measuring the deflection rate w in the downward direction, the radial and circumferential rates of curvature κ, λ are given by

$$\kappa = -\frac{d^2w}{dr^2}, \qquad \lambda = -\frac{1}{r}\frac{dw}{dr}. \tag{46}$$

The moment-rate of curvature relation associated with the yield condition (43) requires

$$\kappa/\lambda = \frac{\partial F}{\partial M}\Big/\frac{\partial F}{\partial N}. \tag{47}$$

From the discussion in Sec. 7, a design compatible with a collapse mode w such that the modified dissipation function $\Delta(w)$ is constant throughout the plate is a minimum volume design. The radial and circumferential strain rates in the upper face of the plate are $-\frac{1}{2}H\kappa$ and $-\frac{1}{2}H\lambda$ respectively and are equal in magnitude and opposite in sign in the lower face. We therefore seek a deflection rate w such that

$$\Delta(w) = D\left(\frac{1}{2}H\frac{d^2w}{dr^2}, \frac{1}{2}\frac{H}{r}\frac{dw}{dr}\right) - \gamma w = \text{constant}, \tag{48}$$

where the form of the dissipation function D is known from the yield condition of the material. At the simply supported edge $r = r_0$ of the plate the deflection rate is zero and at the center $dw/dr = 0$ since λ is finite there. Also for equilibrium $M = N$ at the center of the plate so that $\kappa = \lambda$ at $r = 0$ since the yield condition (43) is symmetrical in M and N for an isotropic material. In general, Eq. (48) will be non-linear and analytical solution of (48) subject to the conditions on w and its derivatives will be difficult. A possible numerical approach is the following. Numerical values are arbitrarily assigned to the positive value of w and the common negative value of d^2w/dr^2 and $(1/r)dw/dr$ at the center,

$$(w)_{r=0} = a > 0, \qquad \left(\frac{d^2w}{dr^2}\right)_{r=0} = \left(\frac{1}{r}\frac{dw}{dr}\right)_{r=0} = -b < 0. \tag{49}$$

The constant on the right hand side of (48) is then determined by substitution of the values (49). With this value of the constant, Eq. (48) is integrated numerically in the range $0 < r \leq r_0$ to give a deflection rate which will be denoted by W. If W_0 is the value of W at $r = r_0$, then the deflection rate w given by

$$w = W - W_0 \tag{50}$$

satisfies Eq. (48) with a different constant on the right hand side and satisfies all the conditions on w and its derivatives.

When the collapse mode w such that $\Delta(w) = \text{constant}$ is known, the rates of curvature

κ, λ can be found from (46). The yield condition (43) and the normality condition (47) then determine M and N in terms of M_0 and hence in terms of the thickness t. The thickness t must be chosen so that M and N satisfy the equilibrium equation (45). At the edge of the plate $M = 0$ so t is taken to be zero at $r = r_0$. Also the resulting equation for t will in general be singular at $r = 0$ and the solution must be such that t is finite at $r = 0$. Thus, in principle, a minimum volume design can be obtained for a material with the general yield condition (43).

The solution for the Tresca yield criterion, which does not define a smooth yield curve, is particularly simple. For the problem considered here the curvature rates κ, λ are non-negative and the condition Δ = constant requires

$$\Delta(w) = -\frac{\sigma_0 H}{2}\left(\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right) - \gamma w = \text{constant}. \tag{51}$$

It follows that the appropriate solution is

$$w = C[J_0(\alpha r) - J_0(\alpha r_0)], \tag{52}$$

where C is a positive constant, $\alpha^2 = 2\gamma/\sigma_0 H$, and J_0 is the Bessel function of the first kind of order zero. The restriction that κ and λ be non-negative requires that $\alpha r_0 < 1.84$ approximately. When $\alpha r_0 > 1.84$ the weight of the material will predominate over its strength.

It remains to determine the thickness t of the faces. As both κ and λ are positive, the Tresca yield criterion requires the bending moments M and N to be positive and equal to the fully plastic moment $M_0 = \sigma_0 H t$. Substitution in the equilibrium equation (45) gives

$$\frac{d^2t}{dr^2} + \frac{1}{r}\frac{dt}{dr} + \alpha^2 t = -\frac{p}{\sigma_0 H}. \tag{53}$$

The appropriate solution of this equation with $t = 0$ at $r = r_0$ where M is zero and t finite at $r = 0$ is

$$t = \frac{p}{\sigma_0 H \alpha^2}\left[\frac{J_0(\alpha r)}{J_0(\alpha r_0)} - 1\right]. \tag{54}$$

As M, N are positive, αr_0 must be less than the first zero of J_0, that is $\alpha r_0 < 2.4$ approximately. For $\alpha r_0 < 1.84$, the design (54) is compatible with the collapse mode (52) for which Δ = constant and is therefore a minimum volume design The design (54) could have been obtained directly from moment considerations alone

As the weight per unit volume γ of the material tends to zero, the design (54) tends to the design for zero body force

$$t = \frac{1}{4}\frac{p}{\alpha_0 H}(r_0^2 - r^2), \tag{55}$$

obtained previously for the von Mises yield condition in Ref. 9.

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BOOK REVIEWS

(*Continued from p. 256*)

Creep and fracture of metals at high temperature. Proceedings of a Symposium held at the National Physical Laboratory, Teddington, England. Her Majesty's Stationery Office, London, and British Information Services, New York, 1956. iv + 420 pp. $5.60.

This book is a summary of the proceedings of an international meeting concerned with the creep and fracture of metals at high temperatures. The meeting was held at the National Physical Laboratory, Teddington, England in May and June 1954. There are twenty-two papers falling into four categories: Section I, Deformation Processes In Simple Materials; Section II, Creep Resistance of Complex Materials; Section III, Theory of Fracture; Section IV, Tertiary Creep and Fracture.

These proceedings provide an excellent summary of a cross section of the present ideas (on an international scale) about creep and fracture at elevated temperatures. In the two years since this conference there have been some significant advances in the understanding of grain boundary motion.

ROHN TRUELL

Introduction to plasticity. By Aris Phillips. The Ronald Press Company, New York, 1956. ix + 230 pp. $7.00.

The first six chapters, which form one-half of the book, deal with the one-dimensional problem of the plastic design of beams and frames, the plastic bending of beams with and without axial forces, the calculation of deflections in beams and frames from the moment-curvature relation, and an elementary theory of curved bars. In the preface the author states that the book is written from the point of view of the stress analyst. Accordingly, approximately twenty pages of Chapters 2-6 are devoted to the listing of results for various cases which may arise in particular problems.

The Prandtl-Reuss stress-strain relations are stated in Chapter 7 and are applied in the remaining two chapters to problems involving combined stresses. Except for a brief discussion of plane strain, the problems treated have the simplifying feature of axial or radial symmetry. The treatment throughout is at a low mathematical level and is at times somewhat inelegant.

R. T. SHIELD

Mathematische Methoden (Volumes I and II of Handbuch der Physik). Edited by S. Flügge. Springer-Verlag, Berlin-Göttingen-Heidelberg. Vol. I: 1956, vii + 364 pp. $17.25; Vol. II: 1955, vii + 520 pp. $20.95.

The volumes contain the following articles. Volume I: Fundamental Concepts of Classical Analysis, Ordinary Differential Equations, Theory of Functions (J. Lense, Technische Hochschule München, 89 pp.)—Partial Differential Equations (J. Lense, 30 pp.)—Elliptic Functions and Integrals (J. Lense, 27 pp.)—Special Functions of Mathematical Physics (J. Meixner, Technische Hochschule Aachen, 71 pp.)—Boundary Value Problems (F. Schlögl, Universität Köln, 135 pp.); Volume II: Algebra (G. Falk, Technische Hochschule Aachen, 116 pp.)—Geometry (H. Tietz, Technische Hochschule Braunschweig, 81 pp.)—Functional Analysis (I. N. Sneddon, University College of North Staffordshire, 151 pp.)—Numerical and Graphical Methods (L. Collatz, Universität Hamburg, 122 pp.)—Modern Computers (H. Bückner, General Electric Company, Schenectady, N.Y., 28 pp.). A critical discussion of these articles, or even a mere listing of section headings, would require more space than is available for this review. Since, however, Sneddon's article, which is in English, may interest a greater number of readers of this Quarterly than the other articles, which are in German, its scope is indicated by the following headings and subheadings: Integration and Abstract Spaces (Introduction—Banach Space)—Integral Transforms (The Laplace Transforms—The Fourier Transforms—The Mellin Transform—The Hankel Transform—Finite Transforms—Approximate Methods of Evaluating Transforms)—Hilbert Space—Schwartz's Theory of Distributions—Variational Methods in Functional Analysis.

(*Continued on p. 320*)

ON THE ANALYTIC FUNCTIONS OF ORDER n[1] (APPLICATION TO THE PLANE ELASTICITY)

BY

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1. In the course of an investigation on a problem of plane elasticity we have been led to consider the so-called "analytic functions of order n". These functions, introduced for the first time by P. Burgatti [2][2], are, by definition, expressions of the form

$$\Omega_n(z, \bar{z}) = \sum_{\nu=0}^{n-1} \bar{z}^\nu f_\nu(z), \tag{1}$$

where $f_\nu(z)$ $(\nu = 0, 1, \cdots, n-1)$ are arbitrary analytic functions of the complex variable $z = x + i\,y$ in a domain D and $\bar{z}$ is the imaginary conjugate[3] of z. The positive integer n indicates the order of the function. The properties of analytic functions of order n seem to be in close connection with those of polyharmonic functions of two variables. On the other hand, the latter, especially the biharmonic functions occur in many problems of physics and mechanics; and therefore, their study is particularly important for applications. In what follows, keeping in view this point, we shall establish certain properties of analytic functions of order n and indicate some applications to two-dimensional elasticity.

An immediate property of the analytic functions of order n is the following.

The product of any finite number of analytic functions of order $n_1, n_2, \cdots, n_m$; $\Omega_{n_1}, \Omega_{n_2}, \cdots, \Omega_{n_m}$ defined respectively in domains $D_1, D_2, \cdots, D_m$ is an analytic function $\Omega_N = \Omega_{n_1}\Omega_{n_2}\cdots\Omega_{n_m}$ of order $N = n_1 + n_2 + \cdots + n_m - (m-1)$ defined in the domain D' which is the common part of $D_1, D_2, \cdots, D_m$.

We shall mention two special cases: (i) the product of an analytic function of order n, Ω_n by an analytic function of order 1, Ω_1, i.e. by an analytic function in the ordinary sense $F(z)$, is an analytic function $\Omega_n^* = F(z)\Omega_n$ of order n; (ii) the mth power of Ω_n is the analytic function $(\Omega_n)^m$ of order $m(n-1)+1$.

The above results follow very readily from the form of expression (1).

2. Connection with real polyharmonic functions. A real polyharmonic function of order h, H of the variables x, y is, by definition, a solution of the equation $(\partial^2/\partial x^2 + \partial^2/\partial y^2)^h H = 0$ or, in terms of the variables z and $\bar{z}$, $\partial^{2h}H/\partial\bar{z}^h\partial z^h = 0$. Burgatti showed that the real and imaginary parts P and Q of an analytic function of order n, Ω_n satisfy the above equations, and called them "conjugate polyharmonic functions"[4]. These

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[2]Numbers in brackets refer to the bibliography at the end of the paper.

[3]In the following any quantity $\bar{x}$ will represent the imaginary conjugate of x.

[4]This designation is justified by the fact that Almansi's representations [1], [2] for two such polyharmonic functions contain two sets of harmonic functions which are conjugate two by two.

functions can be written in the following form

$$\begin{cases} P = \frac{1}{2}(\Omega_n + \overline{\Omega}_n) = \frac{1}{2}\sum_{\nu=0}^{n-1}[\bar{z}^\nu f_\nu(z) + z^\nu \bar{f}_\nu(\bar{z})], \\ Q = \dfrac{1}{2i}(\Omega_n - \overline{\Omega}_n) = \dfrac{1}{2i}\sum_{\nu=0}^{n-1}[\bar{z}^\nu f_\nu(z) - z^\nu \bar{f}_\nu(\bar{z})]. \end{cases} \tag{2}$$

Burgatti's definition is concerned with the conjugate polyharmonic functions of order n which are the real and imaginary parts of an analytic function of the same order n, Ω_n . However, it should be remarked that the number n, i.e. the order of the analytic function Ω_n , is generally different from the order of harmonicity n' of the polyharmonic functions which constitute its real and imaginary parts. It is also clear that the number n' is less than or equal to n, and that n' is the highest power plus 1 of the product $z\,\bar{z}$ existing in the terms of Ω_n . For later reference, we shall call these numbers n and n', "order of analyticity" and "order of harmonicity" respectively of the function Ω_n . As examples we can cite the functions

$$\sum_{\nu=0}^{n-1} a_\nu \bar{z}^\nu, \qquad \sum_{\nu=0}^{n-1} a_\nu \bar{z}^\nu z^\nu$$

which are both of order of analyticity n, and have as order of harmonicity 1 and n respectively. The coefficients a_ν are arbitrary complex constants.

3. Differential relations between two conjugate polyharmonic functions. Let P and Q be two conjugate polyharmonic functions of order m which we assume to be the real and imaginary parts of an analytic function of the same order m, Ω_m ; that is, $\Omega_m = P + iQ$. The equation $\partial^m \Omega_m / \partial \bar{z}^m = 0$ or $(\partial/\partial x + i\partial/\partial y)^m (P + iQ) = 0$ is equivalent to (1). If $\delta_1^{(m)}$ and $\delta_2^{(m)}$ designate respectively the real and imaginary parts of the operator $(\partial/\partial x + i\partial/\partial y)^m$, then we have

$$\delta_1^{(m)} + i\delta_2^{(m)} = (\partial/\partial x + i\partial/\partial y)^m. \tag{3}$$

The expansion of the right-hand side of (3) by the binomial formula furnishes for $\delta_1^{(m)}$ and $\delta_2^{(m)}$ the following expressions:

$$\delta_1^{(m)} = \sum_{\nu=0}^{N} (-1)^\nu \binom{m}{2\nu} \partial^m / \partial x^{m-2\nu}\, \partial y^{2\nu}$$

[$N = (m-1)/2$ if m is odd and $N = m/2$ if m is even],

$$\delta_2^{(m)} = \sum_{\nu=0}^{N} (-1)^\nu \binom{m}{2\nu+1} \partial^m / \partial x^{m-(2\nu+1)}\, \partial y^{2\nu+1}$$

[$N = (m-1)/2$ if m is odd and $N = (m-2)/2$ if m is even], where $\binom{m}{k}$ is the number of combinations of m distinct objects taken k at a time and $\binom{m}{0}$ is assumed to be equal to 1. Then writing $(\delta_1^{(m)} + i\delta_2^{(m)})(P + iQ) = 0$, we obtain the partial differential equations

$$\delta_1^{(m)} P = \delta_2^{(m)} Q, \qquad \delta_1^{(m)} P = -\delta_2^{(m)} Q, \tag{4}$$

which constitute for two arbitrary polyharmonic functions P and Q of the same order m, a necessary condition that they be conjugate or, in other words, that they be the

real and imaginary parts of an analytic function of order m. In this respect the relations (4) correspond to those of Cauchy-Riemann for two conjugate harmonic functions[5].

4. Real analytic functions of order n. We shall next establish the general form of the real analytic functions of order n which occur in certain applications. The problem is to determine functions $g_\nu(z)$ in such a way that the expression

$$\omega_n = \sum_{\nu=0}^{n} \bar{z}^\nu g_\nu(z) \quad .$$

be real. It is known *a priori* that the functions $g_\nu(z)$ are polynomials in z of degree $n-1$ with suitable coefficients. The latter can be calculated in the following way

$$\begin{aligned}
\omega_n = \bar{z}^{n-1}(a_1^1 z^{n-1} &+ a_1^2 z^{n-2} + \cdots + a_1^{n-1} z + a_1^n) \\
+ \cdots \quad & \cdots \quad \cdots \quad \cdots \\
+ \bar{z}^{n-\delta}(a_\delta^1 z^{n-1} &+ a_\delta^2 z^{n-2} + \cdots + a_\delta^n) \\
+ \cdots \quad & \cdots \quad \cdots \quad \cdots \\
+ \bar{z}(a_{n-1}^1 z^{n-1} &+ a_{n-1}^2 z^{n-2} + \cdots + a_{n-1}^n) \\
+ (a_n^1 z^{n-1} &+ a_n^2 z^{n-2} + \cdots + a_n^n),
\end{aligned}$$

or, more compactly,

$$\omega_n = \sum_{\nu=1}^{n} \sum_{\mu=1}^{n} a_\nu^\mu \bar{z}^{n-\nu} z^{n-\mu}, \tag{5}$$

where a_k^k is real and $a_i^j = a_j^k$. The expression (5) is the general form of a real analytic function of order n; it can be observed that this expression is a special polyharmonic function and that the order of analyticity and the order of harmonicity are necessarily equal for such a function.

5. Definition of the transformation T_k. Application to the analytic functions of order n. Let $p(z)$ and $q(z)$ be two polynomials in z both of the same degree k and having no common factor,

$$p(z) = \sum_{\nu=0}^{k} a_\nu z^\nu, \qquad q(z) = \sum_{\nu=0}^{k} b_\nu z^\nu,$$

and replace in an analytic function of order n,

$$\Omega_n = \sum_{\nu=0}^{n-1} \bar{z}^\nu f_\nu(z)$$

z and $\bar{z}$ by p/q and $\bar{p}/\bar{q}$ respectively. This substitution will be called a "transformation T_k". The effect of this transformation on Ω_n is expressed by

Theorem 1. *If $\Omega_n(z, \bar{z}) = \sum_{\nu=0}^{n-1} \bar{z}^\nu f_\nu(z)$ is an analytic function having as order of analyticity and order of harmonicity the same number n, then the expression $(\bar{p})^{n-1}\,\Omega_n(p/q, \bar{p}/\bar{q})$ is an analytic function having as its two orders the same number $k(n-1)+1$.*

Proof. When in $\Omega_n(z, \bar{z})$ the variables z and $\bar{z}$ are replaced by p/q and $\bar{p}/\bar{q}$, respectively

[5] One has also $\nabla^{2m} = \delta_1^{(2m)} + \delta_2^{(2m)}$, for $\nabla^{2m} = (\partial/\partial x + i\partial/\partial y)^m\,(\partial/\partial x - i\partial/\partial y)^m = (\delta_1^{(m)} + i\delta_2^{(m)})(\delta_1^{(m)} - i\delta_2^{(m)}) = \delta_1^{(2m)} + \delta_2^{(2m)}$; and the condition that H be a polyharmonic function of order m takes the form $(\delta_1^{(2m)} + \delta_2^{(2m)})\,H = 0$.

this function becomes

$$\Omega_n(p/q, \bar{p}/\bar{q}) = \sum_{\nu=0}^{n-1} (\bar{p}/\bar{q})^{\nu} f_{\nu}(p/q)$$

$$= 1/(\bar{q})^{n-1} \sum_{\nu=0}^{n-1} \bar{p}^{\nu} \bar{q}^{n-(\nu+1)} f_{\nu}(p/q);$$

hence we obtain

$$(\bar{q})^{n-1} \Omega_n(p/q, \bar{p}/\bar{q}) = \sum_{\nu=0}^{n-1} \bar{p}^{\nu} \bar{q}^{n-(\nu+1)} f_{\nu}(p/q). \tag{6}$$

In (6) the product $\bar{p}^{\nu}\, \bar{q}^{n-(\nu+1)}$ is a polynomial in $\bar{z}$ of degree $k(n-1)$ and $f_{\nu}(p/q)$ is a function $f_{\nu}^{*}(z)$ of z, then the right-hand side of (6) is of the form

$$\sum_{\lambda=0}^{k(n-1)} \bar{z}^{\lambda} F_{\lambda}(z) \tag{7}$$

which represents an analytic function of order of analyticity $k(n-1)+1$. Now writing $F_{\lambda}(z)/z^{\lambda} = G(z)$, (7) becomes

$$\sum_{\nu=0}^{k(n-1)} \bar{z}^{\lambda} z^{\lambda} G(z). \tag{8}$$

From (8) it follows that the new analytic function also has the number $k(n-1)+1$ as well as order of harmonicity. Hence the proof is complete.

Special case. The transformation T_k in general changes the two orders of Ω_n ; these orders remain unchanged if and only if $k = 1$, so that the linear transformation T_1 , $(az + b)/(cz + d)$ where a, b, c, d are arbitrary complex constants, gives from Ω_n another analytic function having the same number n as both orders.

6. Effects of the transformation T_k on a polyharmonic function. Let us now consider a real polyharmonic function of order n of two variables and suppose that it constitutes the real or imaginary part, say real part, of an analytic function of the same order n. The effect of the transformation T_k on such a function is expressed by the following theorem

Theorem 2. *If $H(z, \bar{z})$ is a real polyharmonic function of order n, then $(q\bar{q})^{n-1} H(p/q, \bar{p}/\bar{q})$ is a polyharmonic function of order $k(n-1)+1$.*

Proof. Writing that $H(z, \bar{z})$ is the real part of an analytic function of order n, Ω_n , we have

$$H(z, \bar{z}) = \tfrac{1}{2}(\Omega_n + \overline{\Omega}_n); \tag{9}$$

then replacing in (9) the variables z and $\bar{z}$ respectively by p/q and $\bar{p}/\bar{q}$, and multiplying both members by $(q\bar{q})^{n-1}$, we obtain

$$(q\bar{q})^{n-1} H(p/q, \bar{p}/\bar{q}) = \tfrac{1}{2}\{(\bar{q}^{n-1} \Omega_n(p/q, \bar{p}/\bar{q}))q^{n-1} + (q^{n-1} \overline{\Omega}_n(\bar{p}/\bar{q}, p/q))\bar{q}^{n-1}\}. \tag{10}$$

By Theorem 1, the term $(\bar{q})^{n-1} \Omega_n(p/q, \bar{p}/\bar{q})$ in the right-hand side of (10) is an analytic function having as both orders the number $k(n-1)+1$. The multiplication of this function by q^{n-1} gives another analytic function of the same order. On the other hand, the term $q^{n-1}\, \overline{\Omega}_n(\bar{p}/\bar{q}, p/q)\, (\bar{q})^{n-1}$ is the imaginary conjugate of the preceding analytic function. It follows that the right-hand side of (10) represents a polyharmonic function of order n. This completes the proof.

The special case mentioned above for Theorem 1 holds also for Theorem 2; that is, the linear transformation $(az + b)/(cz + d)$ conserves the order of polyharmonic functions. Theorem 1 may be considered, in the case of two variables, as a generalization of the similar theorems previously established by Lord Kelvin [5], Levi-Civita [4], P. Burgatti [2] and Nicolesco [8].

7. Nets of principal lines attached to analytic functions. Let ρ and -2θ be the absolute value and argument of an analytic function of order n, Ω_n, and write

$$\Omega_n = \rho e^{2i\theta} = \sum_{\nu=0}^{n-1} \bar{z}^\nu f_\nu(z);$$

the values of ρ and θ are given by $\rho = (\Omega_n \overline{\Omega}_n)^{\frac{1}{2}}$ and

$$\theta = \tfrac{1}{2} \operatorname{arc\,tg} i \frac{\Omega_n - \overline{\Omega}_n}{\Omega_n + \overline{\Omega}_n} = \tfrac{1}{4} i \log \frac{\overline{\Omega}_n}{\Omega_n}. \tag{11}$$

If the function Ω_n contains real factors (constant or not), then the value of its argument is evidently independent of those factors and is always equal to that of the complex factor which Ω_n contains. Suppose, for instance, that Ω_n is the product of an analytic function of order r, Ω_r, which contains no real factor, and a real analytic function of order s, ω_s; it is clear, by Eq. (1), that $r = n - (s - 1)$ and that $\Omega_n = \omega_s . \Omega_{n-(s-1)}$. Since ω_s is real, we have $\arg \Omega_n = \arg \Omega_{n-(s-1)}$. In the following, the order of harmonicity h of $\Omega_{n-(s-1)}$, that is the order of harmonicity of the complex factor contained in Ω_n, will also be called "order of the argument of Ω_n", and we shall use the notation $-2\theta_h$ to designate an argument of order h.

The application of the transformation T_k to the argument of an analytic function Ω_n gives a simple property which may be observed as a corollary of one of the preceding theorems.

If $-2\theta_h\ (z, \bar{z})$ is the argument of an analytic function of order h, Ω_h, then $-2\theta_h\ (p/q, \bar{p}/\bar{q})$ is the argument $-2\theta_l$ of an analytic function whose order of harmonicity is less than or equal to $l = k(h - 1) + 1$. The proof of this result is easily obtained when Theorem 2 is applied to both terms of the first expression of θ in (11), or when Theorem 1 is used for the second expression in (11).

We now consider the family of curves (C_h) in the plane (x, y) whose direction angle with the x-axis at each point is equal to θ, -2θ being the argument of an analytic function Ω_n. The curves (C_h) with their orthogonal trajectories constitute an orthogonal net of plane curves which we call "net of principal lines attached to Ω_n". As an example, when $h = 1$, we have an isometric net of plane curves. We shall be concerned with certain properties of nets of principal lines which can easily be deduced from those of analytic functions of order n.

Theorem 3. *Let (C_h) be a net of principal lines attached to an analytic function of order n, Ω_n; it is possible to determine real analytic functions ω_m of arbitrary order m such that the function $\omega_m . \Omega_n$ admits (C_h) as a net of principal lines.*

Proof. Let $\Omega_n = \rho e^{-2i\theta_h} = \sum_{\nu=0}^{n-1} \bar{z}^\nu f_\nu(z)$ be the given analytic function and let

$$\omega_m = \sum_{\nu=1}^{m} \sum_{\mu=1}^{m} a_\nu^k \bar{z}^{m-\nu} z^{m-\mu}$$

be a real analytic function of order m; the argument of the analytic function of order

$n + m - 1$, $\Omega_{n+m-1} = \omega_m \Omega_n$ will be $\arg \Omega_{n+m-1} = \arg \Omega_n = -2\theta_k$; if (C_k) is assumed to be a net of principal lines attached to an analytic function of order $N = n + m - 1$, then

$$\Omega_N = \omega_m \cdot \Omega_n = \Omega_n \cdot \sum_{\nu=1}^{m} \sum_{\mu=1}^{m} \bar{z}^{m-\nu} z^{m-\mu}.$$

Special case. When $n = 1$, we have an isometric net (C_1), and Ω_N is of the form

$$\Omega_N = F(z) \sum_{\nu=1}^{m} \sum_{\mu=1}^{m} a_\nu^\mu \bar{z}^{m-\nu} z^{m-\mu}, \tag{12}$$

where $F(z)$ is an ordinary analytic function whose argument is the harmonic angle -2θ.

We shall end this section with the extension of a property of the ordinary analytic functions to the analytic functions of order n. This will be the subject of the following problem.

Problem. Consider two analytic functions of the same order. Determine the conditions under which the arguments of these functions are two symmetric angles with respect to the x-axis. Let

$$\Omega_{n+1} = \sum_{\mu=0}^{n} \bar{z}^\mu f_\mu(z), \qquad \Omega_{n+1}^* = \sum_{\nu=0}^{n} \bar{z}^\nu f_\nu^*(z)$$

be two analytic functions both of order $n + 1$, and let

$$-\frac{1}{2i}[\log(\bar{\Omega}_{n+1}/\Omega_{n+1})], \qquad -\frac{1}{2i}[\log(\bar{\Omega}_{n+1}^*/\Omega_{n+1}^*)] \tag{13}$$

be their arguments. The condition that the angles (13) be symmetric with respect to the x-axis is equivalent to $\bar{\Omega}_{n+1} \cdot \Omega_{n+1}^* = \Omega_{n+1} \cdot \Omega_{n+1}^*$ = a real quantity. Hence, we have to form the product $\Omega_{n+1}\,\Omega_{n+1}^*$ which will be a real analytic function of order $2n + 1$, ω_{2n+1} (Sects. 1 and 4). Therefore,

$$\omega_{2n+1} = \sum_{\mu=0}^{n} \sum_{\nu=0}^{n} \bar{z}^{\mu+\nu} f_\mu(z) f_\nu^*(z) = \sum_{\mu=1}^{2n+1} \sum_{\nu=1}^{2n+1} a_\nu^\mu \bar{z}^{2n-\nu+1} z^{2n-\mu+1}. \tag{14}$$

Equating now the coefficients of $\bar{z}^m (m = 0, 1, \cdots, 2n)$ in the two members of (14), we have

$$\begin{cases} \sum_{\mu=0}^{m} f_\mu f_{m-\mu}^* = \sum_{\mu=1}^{2n+1} a_{2n-m+1}^\mu z^{2n-\mu+1}, & \text{if} \quad n \geqq m \geqq 0, \\ \text{or} \\ \sum_{\mu=m-n}^{n} f_\mu f_{m-\mu}^* = \sum_{\mu=1}^{2n+1} a_{2n-m+1}^\mu z^{2n-\mu+1}, & \text{if} \quad 2n \geqq m > n. \end{cases} \tag{15}$$

When m varies from 0 to $2n$, (15) furnishes a system of $2n + 1$ equations between $2n + 2$ unknown functions f_μ and f_ν^* $(\mu, \nu = 0, 1, \cdots, n)$. If one takes, for instance, f_μ/f_0 and f_ν^*/f_0^* as unknowns of the problem, the number of the unknowns reduces to $2n$ for $2n + 1$ equations. Suppose that the unknowns f_ν^*/f_0^* are eliminated from the system (15), the n unknowns f_μ/f_0 will remain for n equations. On the other hand, Eq. (15) are obviously symmetric with respect to f_μ and f_ν^* ; then either f_μ/f_0 or f_ν^*/f_0^* will satisfy the same conditions. We do not want to enter into the detail of this discussion. A solution of the problem is known *a priori*; it consists of the isometric case, i.e. when the angle θ

is a harmonic function. In order to indicate an example of a non-isometric case, we shall give the following functions

$$\Omega_{n+1} = F(z) \sum_{k=1}^{n+1} \sum_{i=1}^{n+1} C_{ki}\, \bar{z}^{n-k+1} z^{n-i+1},$$
$$\Omega_{n+1}^{*} = 1/F(z) \sum_{k=1}^{n+1} \sum_{i=1}^{n+1} \overline{C_{ki}}\, \bar{z}^{n-i+1} z^{n-k+1}, \tag{16}$$

where the coefficients C_{ki} are arbitrary complex constants.

8. Applications to plane elasticity. Since the stress deviator of a two-dimensional elastic field is an analytic function of order 2, $\Omega_2 = \bar{z} f_1(z) + f_0(z)$, the above developments on the analytic functions of order n may be used, as far as the geometry of the elastic field is concerned, for the problems of two-dimensional elasticity. The net of the lines of principal stress constitutes a net of principal lines attached to the analytic function Ω_2 which represents the stress deviator of our elastic field; therefore, the theorem established for the nets attached to analytic functions of order n can be applied to the case of plane elasticity.

A. We consider first the special case of Theorem 1 for which $k = 1$. Thus we have:

If $\Omega_2 = \bar{z} f_1(z) + f_0(z)$ is the stress deviator of a two-dimensional elastic field in a domain D, then $(\bar{c}\bar{z} + \bar{d})\cdot \Omega_2[(az + b)/(cz + d), (\bar{a}\bar{z} + \bar{b})/(\bar{c}\bar{z} + \bar{d})]$ is the stress deviator of another two-dimensional elastic field in the domain D' into which D is carried by the transformation T_1.

B. Considering now the special case of transformation T_1 in the property concerning the invariance of the argument of an analytic function of order n, we can state:

Let $\theta(z, \bar{z})$ be the direction angle of a plane elastic stress field defined in a domain D; the angle

$$\theta\left(\frac{az + b}{cz + d}, \quad \frac{\bar{a}\bar{z} + \bar{b}}{\bar{c}\bar{z} + \bar{d}}\right)$$

is the direction angle of another plane elastic stress field defined in the domain D' into which D is carried by the transformation T_1.

C. Let us now take Theorem 3 in the special case of isometric nets. For plane elasticity this theorem takes the following form which states, as a special case of the general property, a well-known result.

Let (C_1) be an isometric net of plane curves defined by means of the argument of an analytic function $\Omega_1 = F(z)$; it is possible to determine real analytic functions of order 2, ω_2, such that the plane elastic stress field defined by the analytic function $\Omega_2 = \omega_2 \Omega_1$ accepts (C_1) as a net of lines of principal stress.

The corresponding analytic function Ω_2 of order 2 is determined by Eq. (12); in fact, taking $m = 2$, $N = 2$ in this formula we obtain

$$\Omega_2 = F(z)(a_1^1 \bar{z} z + a_2^1 z + a_1^2 \bar{z} + a_2^2), \tag{17}$$

where the coefficients have the same meaning as in Sect. 4. This question has previously been discussed in different ways by P. F. Nemenyi [6], U. Wegner [11], W. Prager [9], S. Süray [10], R. Legendre [3] and P. F. Nemenyi and A. W. Saenz [7].

D. The general problem discussed above has been solved in the special case of plane elasticity by Nemenyi and Saenz [7]; these authors also gave an example of a non-

isometric case of this problem. A more general example may be deduced from the expressions (16) if one takes $n = 1$:

$$\Omega_2 = F(z)(a_{11}\bar{z}z + a_{12}\bar{z} + a_{12}z + a_{22}),$$

$$\Omega_2^* = 1/F(z)(\bar{a}_{11}z\bar{z} + \bar{a}_{12}\,z + \bar{a}_{22}),$$

where the coefficients are arbitrary complex constants.

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THE REFLECTION COEFFICIENT OF A SURFACE OF RAYLEIGH DISTRIBUTED IMPEDANCE*

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Abstract. Formulas are obtained for the probability distribution $P(k)$ of the amplitude k of $(\mathbf{z} - c_2)^2 (\mathbf{z} + c_1)^{-2}$ in which c_1 and c_2 are real constants. $\mathbf{z}$ is a complex number of the form $\mathbf{c} + \mathbf{z}_1$ where $\mathbf{c}$ is a complex constant and $\mathbf{z}_1$ is a complex variable of Gaussian distribution of amplitude and uniform distribution of phase (see Fig. 1).

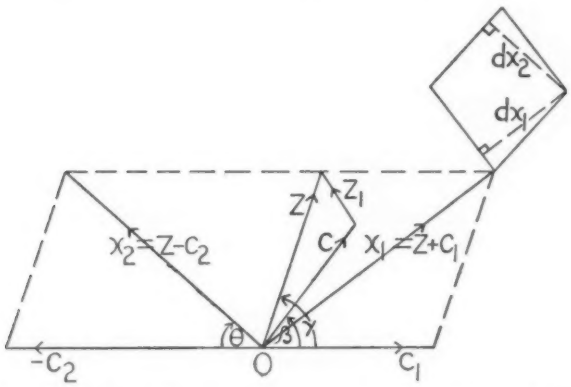

FIG. 1. Geometrical configuration of the vectors $\mathbf{z} + c_1$ and $\mathbf{z} - c_2$. The parallelogram containing dx_1 and dx_2 represents the area swept by the end point of $\mathbf{x}_1$ as x_1 ranges between x_1 and $x_1 + dx_1$ and $\mathbf{x}_2$ ranges between x_2 and $x_2 + dx_2$. The end point of $\mathbf{z}$ ranges over an equal area.

The above relation between k and $\mathbf{z}$ arises when $c_1 = c_2$ in connection with the reflection of a plane wave of sound from a plane surface and relates the power reflection coefficient k to the acoustic impedance ratio $\mathbf{z}$.

The present paper expresses $P(k)$ in terms of the modified Bessel functions I_0 and I_1 if $\mathbf{c}$ is real, and in terms of the incomplete gamma function if $\mathbf{c}$ is complex. Asymptotic formulas are included to allow calculation of $P(k)$ beyond the range of tabulation of the modified Bessel and incomplete gamma functions.

1. Introduction. The concept of surface impedance is widely used to describe the behavior of a surface with regard to the reflection of acoustic or electromagnetic radiation. The normal acoustic impedance is defined to be the ratio of the acoustic pressure at the surface to the particle velocity normal to the surface [1-3]. Division by ρc, where ρ is the density of the propagating medium and c is the sound velocity, results in the acoustic impedance ratio. Similarly the electromagnetic impedance is defined to be the ratio of the tangential electric field at the surface to the tangential magnetic field in the direction at right angles [4, 5, p. 80]. The impedance ratio is then obtained upon division by $(\mu/\epsilon)^{\frac{1}{2}}$ where μ and ϵ are respectively the permeability and dielectric constant of the propagating medium.

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In the acoustical case the velocity potential ψ of the field at the reflecting surface depends upon the impedance ratio ζ through the relation

$$\psi/(\partial\psi/\partial z) = -\zeta/(ik_0), \tag{1}$$

where k_0 is the wave number of the incident radiation and z is the direction of the outward normal to the surface. In the reflection of a plane electromagnetic wave the component of electric field parallel to the surface and at right angles to the direction of incidence satisfies Eq. (1) at the surface. In either instance the impedance ratio is related to the power reflection coefficient k of the surface with regard to a plane wave incident by the relation

$$k = |\, 1 - \zeta \cos\alpha \,|^2 \, |\, 1 + \zeta \cos\alpha \,|^{-2}, \tag{2}$$

where α is the angle of incidence measured from the normal to the reflecting surface. Graphs of the dependence of k upon ζ for various complex values of ζ are given by Morse [2, p. 367].

Suppose that the surface impedance is not uniform but that its variation is insignificant over a distance equal to the wavelength of the incident radiation. Then the reflected radiation received at a distant point is predictable by ray theory, and it consists of a plane wave reflected with a power reflection coefficient given by (2) in which ζ is the impedance ratio of that portion of the surface that may be viewed at the angle of reflection α from the receiver [6]. As the receiver moves in space the variation of received power is a measure of the variation of $|\, 1 - \zeta \cos\alpha \,|^2 \, |\, 1 + \zeta \cos\alpha \,|^{-2}$ over the reflecting surface.

For many non-uniform reflecting surfaces the impedance of any portion cannot be measured directly since the placing of apparatus near the surface affects the local sound or electric field. In general ζ has both real and imaginary components while $(1 - \zeta \cos\alpha)/(1 + \zeta \cos\alpha)$ may often be measured in terms of amplitude only and not with regard to phase. It is consequently of some practical interest to consider the manner in which the probability distribution of amplitude and phase of ζ affects the distribution of $|\, 1 - \cos\alpha \,|^2 \, |\, 1 + \zeta \cos\alpha \,|^{-2}$.

The present paper derives the probability density of the amplitude k of $|\, \mathbf{z} - c_2 \,|^2 \, |\, \mathbf{z} + c_1 \,|^{-2}$ in which c_1 and c_2 are real constants. $\mathbf{z}$ is a complex number of the form $\mathbf{c} + \mathbf{z}_1$ where $\mathbf{c}$ is a complex constant and $\mathbf{z}_1$ is a complex variable of Gaussian distribution of amplitude and uniform distribution of phase. The geometrical configuration of the vectors which represent these complex numbers is as indicated in Fig. 1.

2. General formula for the probability density of k. If $\mathbf{z}$ has Gaussian distribution of amplitude z_1 and uniform distribution of phase θ the joint probability density of z_1 and θ is

$$(1/\pi\sigma^2) z_1 \exp(-z_1^2/\sigma^2) \tag{3}$$

in terms of the standard deviation σ of the amplitude. The factor z_1 appears in (3) since $z_1 dz_1 d\theta$ represents the area swept by the end point of $\mathbf{z}_1$ as z_1 and θ vary between z_1 and $z_1 + dz_1$ and θ and $\theta + d\theta$ respectively.

Denote $\mathbf{z} + c_1$ by $\mathbf{x}_1$ and $\mathbf{z} - c_2$ by $\mathbf{x}_2$ (see Fig. 1). Possible values of $\mathbf{x}_1$ and $\mathbf{x}_2$ are those that permit a real solution z of the equation

$$\begin{aligned} z^2 &= c_2^2 + x_2^2 - 2c_2 x_2 \cos\theta \\ &= c_2^2 + x_2^2 - c_2 c_3^{-1}(c_3^2 + x_2^2 - x_1^2), \end{aligned} \tag{4}$$

where c_3 denotes $c_1 + c_2$ and θ is the angle between $\mathbf{x}_2$ and $-c_2$. Equation (4) has two solutions in z if

$$|x_1 - x_2| < |c_3| < x_1 + x_2 \tag{5}$$

and no solution otherwise. As x_1 varies between x_1 and $x_1 + dx_1$ and x_2 varies between x_2 and $x_2 + dx_2$ the end point of $\mathbf{z}$, and hence of $\mathbf{z}_1$, ranges over an area of

$$\begin{aligned} &4x_1x_2[4x_1^2x_2^2 - (c_3^2 - x_1^2 - x_2^2)^2]^{-\frac{1}{2}}\,dx_1\,dx_2 \\ &\qquad = [4x_1^2x_2^2 - (c_3^2 - x_1^2 - x_2^2)^2]^{-\frac{1}{2}}\,d(x_1^2)\,d(x_2^2). \end{aligned} \tag{6}$$

Thus the joint probability density of x_1^2 and x_2^2 is

$$(1/\pi\sigma^2)[4x_1^2x_2^2 - (c_3^2 - x_1^2 - x_2^2)^2]^{-\frac{1}{2}} \exp(-z_1^2/\sigma^2). \tag{7}$$

Because the element of area (6) includes pairs of solutions of (4) it may be supposed in (7) that the phase of $\mathbf{x}_1$ and $\mathbf{x}_2$ lies between zero and π.

Now put $x_2^2 = kx_1^2$ and $d(x_2^2) = x_1^2\,dk$. The probability density of k is $P(k)$ where

$$P(k) = (1/\pi\sigma^2)\int [4k^2x_1^2 - (c_3^2 - x_1^2 - kx_1^2)^2]^{-\frac{1}{2}}x_1^2 \exp(-z_1^2/\sigma^2)\,d(x_1^2), \tag{8}$$

with integration over the range of x_1^2 from $c_3(1 + k^{\frac{1}{2}})^{-2}$ to $c_3(1 - k^{\frac{1}{2}})^{-2}$.

The substitution $x_1^2 = c_3(1 - k)^{-2}[1 + k - 2k^{\frac{1}{2}}\cos\phi]$ reduces (8) to the form

$$P(k) = [c_3^2\,|\,1 - k\,|^{-3}/\pi\sigma^2]\int_0^{\pi} [1 + k - 2k^{\frac{1}{2}}\cos\phi]\exp(-z_1^2/\sigma^2)\,d\phi. \tag{9}$$

Before proceeding with the evaluation of (9) it is necessary to express z_1 in terms of ϕ. If $\mathbf{c} = c\exp(i\beta)$ and $\mathbf{z} = z\exp(i\gamma)$ then

$$z_1^2 = c^2 + z^2 - 2cz\cos(\gamma - \beta). \tag{10}$$

Let c_2/c_1 be denoted by a. Then

$$\begin{aligned} \cos\gamma &= (x_1^2 - c_1^2 - z^2)(2c_1z)^{-1} \\ &= [a + k - k^{\frac{1}{2}}(1 + a)\cos\phi]c_1(1 - k)^{-1}z^{-1} \end{aligned} \tag{11}$$

and, since γ lies between zero and π,

$$\sin\gamma = k^{\frac{1}{2}}(1 + a)c_1\,|\,1 - k\,|^{-1}z^{-1}\sin\phi. \tag{12}$$

Substitution of the above expressions for $\cos\gamma$ and $\sin\gamma$ into (10) enables z_1^2 to be written in the form

$$\begin{aligned} z_1^2 = c^2 &+ [4ak + (1 + k)(a^2 + k)]c_1^2(1 - k)^{-2} \\ &- 2cc_1(a + k)(1 - k)^{-1}\cos\beta \\ &- 2k^{\frac{1}{2}}(1 + a)c_1(1 - k)^{-2}R\cos(\phi - \omega), \end{aligned} \tag{13}$$

where R and ω are given by

$$R^2 = c^2(1 - k)^2 + c_1^2(a + k)^2 - 2(1 - k)(a + k)cc_1\cos\beta \tag{14}$$

and

$$\cos\omega = [(k + a)c_1 - (1 - k)c\cos\beta]R^{-1}, \tag{15}$$

$$\sin \omega = | 1 - k | cR^{-1} \sin \beta. \tag{16}$$

Thus (9) becomes

$$\begin{aligned} P(k) = {} & [c_3^2 | 1 - k |^{-3}/\pi\sigma^2] \exp (- c^2/\sigma^2) \\ & \times \exp \{- [4ak + (1 + k)(a^2 + k)](1 - k)^{-2}c_1^2/\sigma^2\} \\ & \times \exp [2cc_1(a + k)(1 - k)^{-1}\sigma^{-2} \cos \beta] \\ & \times \int_0^{\pi} (1 + k - 2k^{\frac{1}{2}} \cos \phi) \exp [b \cos (\phi - \omega)]\, d\phi, \end{aligned} \tag{17}$$

where $b = 2k^{\frac{1}{2}} (1 + a)c_1 R(1 - k)^{-2}\sigma^{-2}$.

Reduction of the formula when c is real. In the case that $\mathbf{c}$ is real and positive then $\beta = 0$ and $\omega = 0$ or π. The case of $\mathbf{c}$ negative follows by interchange of $\mathbf{c}_1$ and $\mathbf{c}_2$ but does not arise when $\mathbf{c}$ denotes an acoustic impedance. The integral in (17) may now be expressed in terms of the modified Bessel functions $I_0(b)$ and $I_1(b)$ as $\pi[(1 + k)I_0(b) - 2k^{\frac{1}{2}}I_1(b)]$ [7, p. 181]. The probability density $P(k)$ is thus

$$\begin{aligned} P(k) = {} & [c_3^2 | 1 - k |^{-3}\sigma^{-2}] \exp (- c^2/\sigma^2) \\ & \times \exp \{- [4ak + (1 + k)(a^2 + k)](1 - k)^{-2}c_1^2/\sigma^2\} \\ & \times \{ \exp [2cc_1(a + k)(1 - k)^{-1}\sigma^{-2}]\}[(1 + k)I_0(b) - 2k^{\frac{1}{2}}I_1(b)], \end{aligned} \tag{18}$$

where $b = 2k^{\frac{1}{2}} (1 + a)c_1 [(a + k)c_1 - (1 - k)c] (1 - k)^{-2}\sigma^{-2}$. This latter form of b includes the effect of the cos ω factor which may be $\pm$ 1.

For values of b between 0 and 10 the functions $I_0(b)$ and $I_1(b)$ are tabulated in [8, p. 226]. For larger values of b it is convenient to make use of an asymptotic form for $P(k)$. Such is necessary, for example, in order to evaluate $P(1)$. For large values of b the asymptotic formula for $I_r(b)$ [7 p. 203] may be applied in the form

$$I_r(b) \sim (2\pi b)^{-\frac{1}{2}}(\exp b)[1 - (4r^2 - 1)/8b + (4r^2 - 1)(4r^2 - 9)/128b^2], \tag{19}$$

when b is positive and through the relation $I_r(b) = (-1)^r I_r(-b)$ when b is negative. The resulting asymptotic expansion for $(1 + k)I_0(b) - 2k^{\frac{1}{2}}I_1(b)$ is

$$(1 + k)I_0(b) - 2k^{\frac{1}{2}}I_1(b) = | 2\pi b |^{-\frac{1}{2}}(1 - k^{\frac{1}{2}})^2(\exp | b |)F(k, b), \tag{20}$$

where

$$\begin{aligned} F(k, b) \sim {} & 1 - 3/8 | b | - 15/128b^2 + (1 + k)(1 - k^{\frac{1}{2}})^{-2}/2 | b | \\ & + 3(1 + k)(1 - k^{\frac{1}{2}})^{-2}/16b^2. \end{aligned} \tag{21}$$

In the above equations $k^{\frac{1}{2}}$ denotes the positive square root of k when b is positive and the negative square root when b is negative.

Substitution of (20) into (17) permits $P(k)$ to be written as

$$\begin{aligned} P(k) = {} & c_3^2\sigma^{-2} | 2\pi b |^{-\frac{1}{2}}(1 - k^{\frac{1}{2}})^2 | 1 - k |^{-3}F(k, b)[\exp (- c^2/\sigma^2)] \\ & \times \{\exp [- (a - k^{\frac{1}{2}})^2(1 + k^{\frac{1}{2}})^{-2}c_1^2/\sigma^2]\}\{\exp [2(a - k^{\frac{1}{2}})(1 + k^{\frac{1}{2}})^{-1}c_1c/\sigma^2]\}. \end{aligned} \tag{22}$$

The equation (22) applies in particular when $k = 1$ and reduces to

$$\begin{aligned} P(1) = {} & (1/8)\pi^{-\frac{1}{2}}(c_3^2/c_1\sigma)(1 + a)^{-1}[\exp (- c^2/\sigma^2)]\{\exp [- \tfrac{1}{4}(1 - a)^2c_1^2/\sigma^2]\} \\ & \times \{\exp [(a - 1)c_1c/\sigma^2]\}(1 + 2\sigma^2/c_3^2). \end{aligned} \tag{23}$$

For sufficiently large values of c it is permissible to put $b = -2k^{\frac{1}{2}}(1 + a)(1 - k)^{-1} c_1 c/\sigma^2$ and hence to write (22) in the form

$$P(k) = \tfrac{1}{2}\pi^{-\frac{1}{2}} c_2^2 \sigma^{-1} [(1 + a)k^{\frac{1}{2}} c_1 c]^{-\frac{1}{2}} \mid 1 - k \mid^{-\frac{1}{2}} \mid 1 + k^{\frac{1}{2}} \mid^{-2} F(k, b)\ []\{\}\{\} \tag{24}$$

in which the final brackets are as in (22). It is to be noted that if $k > 1$ then b is positive and $P(k)$ contains the factor $\mid 1 - k^{\frac{1}{2}} \mid^{-2}$. However, if $k < 1$ then b is negative and $P(k)$ contains the factor $\mid 1 - k^{\frac{1}{2}} \mid^{-5/2}$. Thus for large c there is greater probability of finding a value of k slightly less than unity than of finding a value slightly in excess of unity. This is in agreement with the fact that $c - c_2 < c + c_1$.

For sufficiently large values of c/σ and c/c_1 the value of b is large compared to unity and the order of magnitude of $P(k)$ in (22) is governed by the product of the exponential terms. The dominant term is $\exp(-c^2/\sigma^2)$ and the product is negligible save for values of k near to the solution of the equation

$$2(a - k^{\frac{1}{2}})(1 + k^{\frac{1}{2}})^{-1} c_1 c/\sigma^2 - (a - k^{\frac{1}{2}})^2 (1 + k^{\frac{1}{2}})^{-2} c_1^2/\sigma^2 = c^2/\sigma^2. \tag{25}$$

The solution of (25) is

$$k = (c - ac_1)^2 (c + c_1)^{-2}$$

which is also the discrete value of k obtained when $\sigma = 0$.

4. Reduction of the formula when c is complex. For complex values of **c** the integrals appearing in (17) cannot be evaluated in a closed form, and for purposes of numerical calculation it is desirable to express them as rapidly convergent series. The required integrals may be written as follows

$$\int_0^{\pi} \exp[b \cos(\phi - \omega)]\, d\phi = \int_0^{\omega} \exp(b \cos\phi)\, d\phi + \int_0^{\pi - \omega} \exp(b \cos\phi)\, d\phi \tag{26}$$

and

$$\int_0^{\pi} \cos\phi \exp[b \cos(\phi - \omega)]\, d\phi = \cos\omega \int_0^{\omega} \cos\phi \exp(b \cos\phi)\, d\phi + \cos\omega \int_0^{\pi - \omega} \cos\phi \exp(b \cos\phi)\, d\phi - (2/b) \sin\omega \sinh(b \cos\omega). \tag{27}$$

Following the method of Hadamard [7, p. 204] put $t = 2b \sin^2(\frac{1}{2}\phi)$ and let b_1 denote $2b \sin^2(\frac{1}{2}\omega)$. Then

$$\begin{aligned} \int_0^{\omega} \exp(b \cos\phi)\, d\phi &= \frac{\exp b}{(2b)^{\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2 2^{3n} b^n} \int_0^{b_1} t^{n-\frac{1}{2}} \exp(-t)\, dt \\ &= \frac{\exp b}{(2b)^{\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2 2^{3n} b^n} \gamma(n + \tfrac{1}{2}, b_1), \end{aligned} \tag{28}$$

where $\gamma(n + \frac{1}{2}, b_1)$ denotes the incomplete gamma function. Similarly

$$\int_0^{\omega} \cos\phi \exp(b \cos\phi)\, d\phi = -\frac{\exp b}{(2b)^{\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2 2^{3n} b^n} \frac{2n + 1}{2n - 1} \gamma(n + \tfrac{1}{2}, b_1). \tag{29}$$

Thus the integrals (26) and (27) may be expressed in the form

$$\int_0^\pi \exp\,[b\,\cos\,(\phi - \omega)]\,d\phi = \frac{\exp\,b}{(2b)^{\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2 2^{3n} b^n} [\gamma(n + \tfrac{1}{2}, b_1) + \gamma(n + \tfrac{1}{2}, b_2)] \tag{30}$$

and

$$\int_0^\pi \cos\phi \exp\,[b\,\cos\,(\phi - \omega)]\,d\phi = -(\cos\omega)\frac{\exp\,b}{(2b)^{\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2 2^{3n} b^n} \frac{2n+1}{2n-1} [\gamma(n + \tfrac{1}{2}, b_1) + \gamma(n + \tfrac{1}{2}, b_2)] - (2/b)\sin\omega \sinh\,(b\,\cos\omega), \tag{31}$$

where $b_2 = 2b\,\cos^2(\frac{1}{2}\omega)$.

The function $\gamma(n + \frac{1}{2}, b_1)$ may be found from Ref. [9] over a wide range of values of n and b_1 . Although b^n appears in the denominators of (30) and (31) the series are convergent for small values of b since $\gamma(n + \frac{1}{2}, b_1)$ tends to zero as $b_1^{n+\frac{1}{2}}/(n + \frac{1}{2})$. For large values of b_1 or b_2 the value of γ may be found from the asymptotic formula [10, p. 135]

$$\gamma(n + \tfrac{1}{2}, b_1) \sim \Gamma(n + \tfrac{1}{2}) - b_1^{n-\frac{1}{2}} \exp\,(-b_1). \tag{32}$$

When b_1 and b_2 are both large compared to unity the first terms in the series (30) and (31) may be combined with (32) to give

$$\int_0^\pi \exp\,[b\,\cos\,(\phi - \omega)]\,d\phi \sim 2\pi(2\pi b)^{-\frac{1}{2}} \exp\,b \tag{33}$$

$$\int_0^\pi \cos\phi \exp\,[b\,\cos\,(\phi - \omega)]\,d\phi \sim (\cos\omega)2\pi(2\pi b)^{-\frac{1}{2}} \exp\,b \tag{34}$$

and hence from (17)

$$P(k) \sim [c_3^2 \mid 1 - k \mid^{-3}/\sigma^2] \exp\,(-c^2/\sigma^2) \times \exp\,\{-[4ak + (1+k)(a^2+k)](1-k)^{-2}c_1^2/\sigma^2\} \times \exp\,[2cc_1(a+k)(1-k)^{-1}\sigma^{-2}\cos\beta] \times 2(2\pi b)^{\frac{1}{2}}(\exp\,b)[1 + k - 2k^{\frac{1}{2}}\cos\omega]. \tag{35}$$

5. Concluding remarks. For given values of the parameters c_1, c_2, $\mathbf{c}$, σ, and β, as k ranges from zero to unity, the range of b is from zero to infinity. Thus, in general, in order to plot the graph of $P(k)$ it is convenient to use Eqs. (30) and (31) when k is small and Eq. (35) when k is close to unity. The case of $k > 1$ has no physical meaning when R in (2) denotes a reflection coefficient. It may be remarked, however, that the quantity $\mid \mathbf{z} - c_2 \mid^2 \mid \mathbf{z} + c_1 \mid^{-2}$ is also related to the problem of the detection of a constant signal c_2 upon a noise background $\mathbf{z}$. It represents the ratio of the power of the signal minus noise to the power of a multiple of the signal plus the same noise. In this instance there is no physical restriction to values of k less than unity.

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—NOTES—

ONE-DIMENSIONAL DIFFUSION WITH THE DIFFUSION COEFFICIENT A LINEAR FUNCTION OF CONCENTRATION: REDUCTION TO AN EQUATION OF THE FIRST ORDER*

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Abstract. The problem considered by Stokes[1], of one-dimensional diffusion from an initially sharp boundary between two semi-infinite columns of liquid, the diffusion constant being a linear function of concentration, is discussed. It is shown how the differential equation may be reduced to an equation of the first order. Some properties of the solution are investigated, and the method of obtaining numerical solutions is considered.

The differential equation to be satisfied is

$$2\frac{d}{dy}\left(D\frac{dc}{dy}\right) + y\frac{dc}{dy} = 0, \tag{1}$$

with boundary conditions

$$c \to c_1 \text{ when } y \to -\infty,$$

$$c \to c_2 \text{ when } y \to \infty,$$

where c is the concentration, c_1 and c_2 the initial concentrations of the two columns, D the diffusion coefficient, and $y = xt^{-\frac{1}{2}}$, where x is the distance from the boundary in the direction of diffusion, and t is the time (see Stokes, loc. cit.). Let D^* be the mean concentration, and let b = ratio of diffusion coefficient at concentration c_1 to that at c_2 ($b \neq 1$). Then, since we assume that D is a linear function of c, we may write

$$D = D^*\left[1 + \frac{1-b}{1+b}\left\{\frac{\frac{1}{2}(c_1+c_2) - c}{\frac{1}{2}(c_1-c_2)}\right\}\right], \tag{2}$$

and, making this substitution in (1), and also writing $y = 2D^{*\frac{1}{2}}.\ z$, the equation to be satisfied becomes

$$\left[1 + \frac{1-b}{1+b}\left\{\frac{\frac{1}{2}(c_1+c_2) - c}{\frac{1}{2}(c_1-c_2)}\right\}\right]\frac{d^2c}{dz^2} - \frac{1-b}{1+b}\frac{1}{\frac{1}{2}(c_1-c_2)}\left(\frac{dc}{dz}\right)^2 + 2z\frac{dc}{dz} = 0, \tag{3}$$

with the conditions $c \to c_1$ when $z \to -\infty$, $c \to c_2$ when $z \to \infty$.

Let now

$$z = \frac{1}{(1+b)^{1/2}}ue^{-u^2} \tag{4}$$

$$c = c_2 + \frac{c_1 - c_2}{1-b}(1 - e^{-2u^2}). \tag{5}$$

*Received March 12, 1956.

[1]R. H. Stokes, Transactions of the Faraday Society, No. 358, Vol. 48, Part 10, October, 1952.

Then, substituting these expressions in (3), the equation to be satisfied reduces to

$$d^2w/du^2 + u\,dw/du - 2(1+u^2)(dw/du)^2 + u(3+u^2)(dw/du)^3 = 0, \tag{6}$$

or finally, writing $p = -dw/du$,

$$dp/du + up + 2(1+u^2)p^2 + u(3+u^2)p^3 = 0, \tag{7}$$

an equation of the first order. It is clear from (5) that $c = c_1$ when $w = \log(b^{-\frac{1}{2}})$ and $c = c_2$ when $w = 0$. Consequently, from (4) and (5) the new boundary conditions are

$$w \to \log(b^{-\frac{1}{2}}) \text{ when } u \to -\infty,$$

$$w \to 0 \text{ when } u \to \infty.$$

We may remark here for reference later on, that we have

$$\begin{aligned} \frac{-1}{c_1 - c_2}\frac{dc}{dz} &= \frac{-2(1+b)^{1/2}}{1-b}e^{-w}\frac{dw/du}{1-u\,dw/du} \\ &= \frac{2(1+b)^{1/2}}{1-b}e^{-w}\frac{p}{1+up} \end{aligned} \tag{8}$$

Now the equation (7) is an ordinary differential equation of the first order and the normal form. Its solutions depend upon one arbitrary constant, which may conveniently be taken to be the value of p when $u = 0$. It follows from the form of (7) that p cannot be zero for any real u. For if $p = 0$ when $u = u_0$, say, then from the form of (7) this integral is certainly analytic at the point $u = u_0$, and is therefore simply the integral $p \equiv 0$. p would therefore be identically zero for all u, which makes w, and hence c, constant throughout, a trivial solution which we reject. We may thus suppose p positive throughout: for a solution for which p is negative throughout is obtained by writing $-p'$ for p and $-u'$ for u, which does not change the form of (7). Suppose, therefore, that $p = \lambda > 0$ when $u = 0$. Then the following results, the proofs of which are given in the appendix, may be established:

(i) Every point of the real axis of u is a regular point of the integral.

(ii) When u increases from zero to ∞, p decreases strictly, and $p \to 0$ as $u \to \infty$, while up increases strictly from zero to a single maximum at a point in the range $0 < u < 1$, this maximum value being $< 1/3$, and then decreases strictly, and $\to 0$ as $u \to \infty$.

(iii) When u decreases from zero to $-\infty$, p first increases strictly to a maximum at a point in the range $-1 \leq u < 0$, then decreases strictly, and $\to 0$ when $u \to -\infty$. But, according to the value of λ, either up decreases strictly from zero to a single minimum value, this value being > -1, at some point for which $u < -1$, then increases strictly, and $\to 0$ as $u \to -\infty$; or else up decreases strictly whenever $u \leq 0$, and $\to -1$ as $u \to -\infty$. For all small enough λ, the former circumstance obtains.

(iv) $\int_0^\infty p\,du$ and $\int_0^\infty up\,du$ are convergent, uniformly with respect to λ, in any positive range of λ.

(v) $\int_{-\infty}^0 p\,du$ and $\int_{-\infty}^0 up\,du$ are convergent uniformly with respect to λ, if λ lies in a range such that $up \to 0$ when $u \to -\infty$.

Suppose then that we select a positive value of λ, and calculate numerically the solution of (7) for which $p = \lambda$ when $u = 0$. It is essential that the calculations be carried on far enough, in the negative direction, to make sure that up passes its (negative)

minimum value, and thereafter increases towards zero; for otherwise we cannot be sure that $|up|$ tends to zero and not to unity as $u \to -\infty$. If this condition be not satisfied, we simply start again with a smaller value of λ: for it is proved in the appendix that for all λ up to a positive limit, $up \to 0$ as $u \to -\infty$. With this understanding, we have $dw/du = -p$, and hence $w = A - \int_0^u p\,du$. But since we require that $w \to 0$ when $u \to \infty$, and since the integral is convergent, we see at once that this gives

$$w = \int_u^\infty p\,du, \tag{9}$$

and thus w is known to any desired degree of accuracy, from the calculated solution of (7). This latter being, by hypothesis, such that $\int_{-\infty}^u p\,du$ is convergent when $u \to -\infty$, the second boundary condition shows that

$$b = \exp\left(-2\int_{-\infty}^\infty p\,du\right), \qquad (b < 1 \text{ since } p > 0). \tag{10}$$

The constant b is therefore calculated from (10), and clearly depends only on λ. From (4) and (5) we then have the required solution parametrically, in terms of u, in the form

$$z = \frac{1}{(1+b)^{1/2}}\, u \exp\left(-\int_u^\infty p\,du\right), \tag{11}$$

$$\frac{c - c_2}{c_1 - c_2} = \frac{1 - \exp\left(-2\int_u^\infty p\,du\right)}{1 - b}, \tag{12}$$

b being given by (10): and from the form of these expressions and from (10), we see that $c \to c_1$ when $u \to -\infty$ and $z \to -\infty$, $c \to c_2$ when $u \to \infty$, $z \to \infty$, so that the original boundary conditions are satisfied, and we have the required solution of (1).

We also have, from (4), (8), (9), and (10)

$$\frac{-1}{c_1 - c_2}\, z\frac{dc}{dz} = \frac{2}{1-b}\, e^{-2w}\, \frac{up}{1+up} = \frac{2}{1-b}\,\frac{up}{1+up}\exp\left(-2\int_u^\infty p\,du\right) \tag{13}$$

and since $1 + up > 0$ throughout[2], and $up \to 0$ when $u \to \pm\infty$, we see that $zdc/dz \to 0$ when $z \to \pm\infty$.

The family of curves of c against z, or the gradient curves $-(c_1 - c_2)^{-1}dc/dz$ against z, can then be plotted for various values of b, by means of the expressions (11), (12), (13), which give these expressions in terms of the parameter u. The method is firstly to calculate the solutions of (7) for an arbitrary positive λ; then to calculate b from (10); and finally z, c and dc/dz in terms of u, from (11), (12), (13).

The advantages of the foregoing method of procedure are two-fold. First of all, there is the advantage of calculating a solution for a first-order differential equation instead of a second-order one. Secondly, the single arbitrary quantity is the value of p at the point $u = 0$, or $z = y = x = 0$. Thus the calculations are begun at a point corresponding to the boundary between the two columns of liquid, and proceed symmetrically in both directions. Thirdly, the convergence of the integrals required to give b, and w in terms of u, has been shown to be uniform with respect to λ when λ does not exceed a fixed

[2]See Appendix.

limit, H say. And denoting the solution of (7) which takes the value λ when $u = 0$ by $p(u, \lambda)$, it is proved in the appendix that $p(u, \lambda) < p(u, \mu)$ when $\lambda < \eta$. Then if $0 < u_1 < u_2$ and $u_3 < u_4 < 0$, we have

$$0 < \int_{u_1}^{u_2} p(u, \lambda)\, du < \int_{u_1}^{u_2} p(u, H)\, du,$$

and

$$0 < \int_{u_3}^{u_4} p(u, \lambda)\, du < \int_{u_3}^{u_4} p(u, H)\, du.$$

It follows that, when calculating the infinite integrals which occur in (10)-(13), if we carry the calculations far enough in each direction to attain the required degree of accuracy for $\lambda = H$, then carrying the calculations the same distance will certainly give the required accuracy for any smaller value of λ.

I am very much indebted to Professor Stokes who drew my attention to this interesting problem.

Appendix

We require to prove the properties (i)-(v), listed above, of that solution of (7) which takes the value $\lambda > 0$ when $u = 0$.

(i) The equation (7) has the fixed singularities $u = 0$, $u = \pm 3^{\frac{1}{2}}i$. The integral such that $p = \lambda > 0$ when $u = 0$ is analytic when $u = 0$. Consequently, it follows from a result of Goursat[3], that a point $u = u_0$ on the real axis can only be a singularity if $p \to \infty$ when $u \to u_0$. We have shown that $p > 0$ throughout. But from (7) it follows that for all large enough positive p, $dp/du < 0$ when $u_0 > 0$ and $| u - u_0 |$ is small, and $dp/du > 0$ when $u_0 < 0$ and $| u - u_0 |$ is small. This contradicts either of the hypotheses that $u_0 > 0$, $p \to +\infty$ when $u \to u_0$ from below, or that $u_0 < 0$, $p \to +\infty$ when $u \to u_0$ from above. It follows that every point of the real axis is a regular point of the integral.

(ii) From (7), when $u > 0$ and $p > 0$, $dp/du < 0$, and p decreases strictly. But $p > 0$ throughout, therefore $p \to \alpha \geq 0$ when $u \to +\infty$. If $\alpha > 0$, clearly $dp/du \to -\infty$, contradicting the hypothesis that $p \to \alpha$. Hence $p \to 0$ as $u \to \infty$.

Next, let $v = up$. Then, using (7), we have, after simplifying,

$$u\, dv/du = v(1 + v)\{(1 - u^2) - (3 + u^2)v\}. \tag{14}$$

When $u > 0$, $v > 0$, and therefore, from (14), when $u \geq 1$, $dv/du < 0$. It follows that when $u \to \infty$, $v \to \beta \geq 0$. But if $\beta > 0$, $dv/du \sim -\beta(1 + \beta)^2\, u \to -\infty$, contradicting $v \to \beta$. Therefore, $v \to 0$ as $u \to \infty$. To show that v has only one maximum, occurring when $0 < u < 1$, we first observe that when $u = 0$, $dv/du = p_0 = \lambda > 0$. When $dv/du = 0$ and $u > 0$, we have, from (14), $d^2v/du^2 = -2v(1 + v)^2 < 0$. Hence there is only one maximum value. At this value, from (14), we see that

$$v_{\max} = (1 - u^2)/(3 + u^2) = 1/3 - 4u^2/3(1 + u^2) < 1/3.$$

Thus proposition (ii) is established.

(iii) When $u < 0$, we may write the equation (7) in the form

$$dp/du = -\frac{p}{u(3 + u^2)}\,[\{u(3 + u^2)p + (1 + u^2)\}^2 + (u^2 - 1)], \tag{15}$$

[3]E. Goursat, "Mathematical Analysis", Vol. II, Part II, sect. 67, pp. 182-184 (Ginn and Co., 1945).

and from this we see that when $u < -1$, $dp/du > 0$. Since $p > 0$, it follows that $p \to \gamma \geq 0$ when $u \to -\infty$. If $\gamma > 0$, $dp/du \to \infty$ as $u \to -\infty$, contradicting $p \to \gamma$. Therefore $p \to 0$ as $\to -\infty$. Again, when $u = 0$, $dp/du = -2\lambda^2 < 0$. There is thus a maximum value between $u = 0$ and $u = -1$. To show that there is only one maximum, we observe from (15) above that when $dp/du = 0$ and $-1 \leq u < 0$,

$$d^2p/du^2 = -p\{1 + 4up + 3(1 + u^2)p^2\}$$

$$= -\frac{p}{3(1 + u^2)}[\{3(1 + u^2)p + 2u\}^2 + 3 - u^2] < 0:$$

and the first part of the proposition is proved.

To establish the second part, we again write $v = up$, so that v again satisfies (14) above, and if $dv/du = 0$ and $u < 0$, we have $d^2v/du^2 = -2v(1 + v)^2$. But when $u < 0$, $v < 0$, and it follows from the form of (14) that $v > -1$ for any $u < 0$: if $v = -1$ when $u = u_0 < 0$, the integral is certainly analytic at u_0, and therefore $v \equiv -1$, $p \equiv -1/u$, a solution which we may reject, since by hypothesis $p = \lambda$ when $u = 0$. Consequently, when $u < 0$, $-1 < v < 0$, and therefore if $dv/du = 0$, $d^2v/du^2 > 0$. Hence there can be at most one negative minimum value of v. But from (14) we see that when $-1 \leq u < 0$, $dv/du > 0$. Hence the minimum, if it exists, must occur for $u < -1$.

Since $dv/du > 0$ when $u = 0$, and since $-1 < v < 0$ when $u < 0$, it follows that when $u \to -\infty$, v either decreases strictly to a limit δ where $-1 \leq \delta < 0$, or else decreases to a minimum and then increases strictly to a limit ϵ, where $-1 < \epsilon \leq 0$. In either case, suppose v tends to a limit η, $-1 < \eta < 0$. Then $dv/du \sim -\eta(1 + \eta)^2 u \to -\infty$ as $u \to -\infty$, contradicting the hypothesis that $v \to \eta$. Hence $\eta = 0$ or else $\eta = -1$. In other words, as u decreases from 0 to $-\infty$, either v decreases to a minimum value between 0 and -1, for some value of u less than -1, and then increases strictly to the limit zero, or else v decreases strictly to the limit -1.

We have yet to show that, for small enough λ, the former event takes place. Since dv/du can have at the most one zero when $u < 0$, and since $dv/du > 0$ when $u = 0$, it follows that if $dv/du < 0$ for any negative u, then certainly v has a minimum, and increases to zero as $u \to -\infty$. But, selecting any fixed negative number N, where $N < -1$, and observing that $p \equiv 0$ is a solution of (7), it is a standard result[4] that a positive number L exists (depending upon N), such that whenever $0 < \lambda \leq L$, the solution of (7) for which $p = \lambda$ when $u = 0$ is such that when

$$u = N, \qquad 0 < p(N) < \frac{1}{-N}\frac{N^2 - 1}{N^2 + 3},$$

and therefore $0 > (3 + N^2)v > 1 - N^2$. From (14) we see, therefore, that whenever $\lambda \leq L$, $dv/du < 0$ when $u = N$: and the result follows.

(iv) Denoting the integral of (7) which takes on the value λ when $u = 0$ by $p(u, \lambda)$, we first observe that if $\lambda < \mu$, then for all u, $p(u, \lambda) < p(u, \mu)$. For by the usual existence theorem, there cannot be two integrals taking a given value for any given value of u, so that $\{p(u, \mu) - p(u, \lambda)\} \neq 0$. But it is impossible for $\{p(u, \mu) - p(u, \lambda)\}$ to be negative for any u: for being continuous, and positive when $u = 0$, it would then be zero for some other u.

[4]E. Goursat, "Cours d'Analyse Mathematique", tome III, sect. 461, pp. 16-18. (Gauthier-Villars, 1942).

Now, integrating the equation (7) between the limits 0 and u, we have

$$\lambda - p = \int_0^u \{up(1 + up)^2 + (2p^2 + 3up^3)\}\, du. \tag{16}$$

Suppose that $0 < \lambda \leq M$. Then we have proved that, for any λ, $up \to 0$ as $u \to \infty$, and that $p(u, \lambda) < p(u, M)$. Hence $p = o(1/u)$, $p^2 = o(1/u^2)$, $up^3 = o(1/u^3)$: and therefore $\int_0^u (2p^2 + 3up^3)du$ is uniformly convergent as $u \to \infty$. But since $p \to 0$ and $p(u, \lambda) < p(u, M)$ as $u \to \infty$, the left hand side of (16) is uniformly convergent to λ. Therefore, since for all large enough u, $(1 + up) > \frac{1}{2}$, it follows at once that $\int_0^u up\, du$ is uniformly convergent when $u \to \infty$: and the same is therefore true of $\int_0^\infty p\, du$.

(v) We may write (16) in the form

$$\lambda - p = -\int_u^0 \{up(1 + up)^2 + (2p^2 + 3up^3)\}\, du.$$

We have proved that for all small enough λ, say $\lambda \leq H$, $up \to 0$ as $u \to -\infty$. In this event, it follows, by precisely the reasoning of (iv) above, that $\int_{-\infty}^0 up\, du$ and $\int_{-\infty}^0 p\, du$ are uniformly convergent.

AN ELEMENTARY SOLUTION OF TWO STRESS CONCENTRATION PROBLEMS IN THE NEIGHBOURHOOD OF A HOLE*

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1. Introduction. The problem of the stress concentration around a central circular hole in a strip in tension (or in bending in its plane) has been investigated extensively. The theoretical solution of this problem has been given by Howland [1]. However, Howland's numerical calculations were restricted to hole diameters which do not exceed half the strip width, and his method requires a rapidly increasing numerical effort for larger holes. Experimental investigations have yielded valuable data on the stress concentration factor for large holes. On the basis of these experimental results Heywood [2] conjectures that the stress concentration factor, referred to the nominal stress in the net section, tends to the value 2 if the hole diameter approaches to the strip width. This conjecture is confirmed by the elementary analysis in Sec. 2.

Recently the more complicated problem of a central spherical hole in a cylindrical bar in tension has been investigated by Ling [3]. His numerical evaluation has also been restricted to a hole radius equal to 1/4 and 1/2 times the bar radius. Here again the numerical effort, required in evaluation of the theory, increases rapidly with increasing hole radius. According to Ling "the value 1 for the stress concentration factor K (referred to the nominal stress in the net section) in the limiting case $\lambda = 1$ (where λ is the ratio of hole radius to bar radius) can be visualized readily from physical considerations of the cylinder" [3, p. 391], and the graph of his results [3, Fig. 2] has been completed for the entire range $0 < \lambda < 1$. However, Ling's argument, which has been cited above, is incorrect, as will be shown in Sec. 3. The correct limiting value of the stress concentration

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factor is

$$\lim_{\lambda \to 1} K = \frac{(6 - 4\nu)(1 + \nu)}{5 - 4\nu^2}, \tag{1}$$

where ν is Poisson's ratio. Ling's graph for the stress concentration factor K as a function of λ should be corrected accordingly.

Our elementary solutions for the stress concentration problems for the strip with a large circular hole and the cylindrical bar with a large spherical hole are based on beam theory and shell theory respectively. It might be objected that both these theories are approximate in character. On the other hand, it may be contended that the accuracy of beam and shell theories increases indefinitely when the "slenderness" of beam and shell increase indefinitely, and the limiting values for the stress concentration factor K for $\lambda \to 1$ should therefore be correct.

Nevertheless, a more rigorous investigation is considered worthwhile, the more so because it may then be possible to investigate the manner in which the limiting value is approached. A suitable method is mentioned in Sec. 4 and some preliminary results for the strip problem, confirming the result of beam theory, are given.

2. Strip with a large circular hole. The strip with a *large circular hole* ($\delta \ll R$) is pictured in Fig. 1, and the upper part to the right of the minimum section is given on a

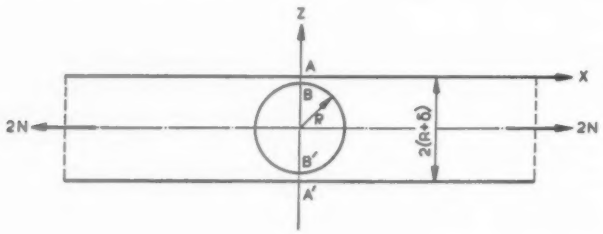

FIG. 1

larger scale in Fig. 2. It is obvious from symmetry that no shear force is transmitted by the minimum cross-section AB. On the other hand, in order to maintain this cross-section in a vertical position, which is again obviously necessary on account of symmetry, a positive bending moment M_0 must occur in this cross-section.

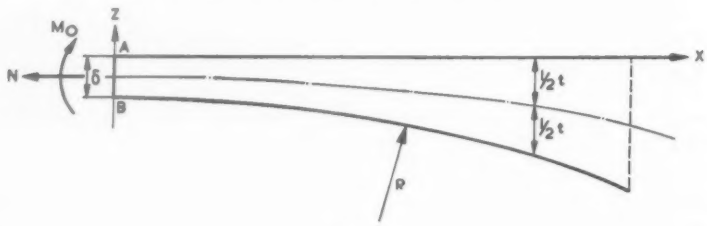

FIG. 2

An element of the "beam" in Fig. 2 has been drawn in Fig. 3. Obviously the shear force is zero and the normal force has the constant value N.

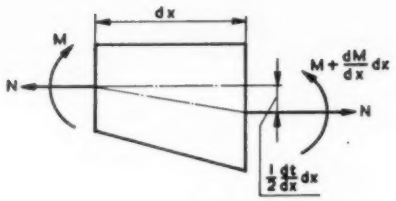

FIG. 3

Equilibrium of moments requires

$$\frac{dM}{dx} = -\frac{1}{2} N \frac{dt}{dx}. \tag{2}$$

In the neighbourhood of the minimum cross-section $x = 0$ the height t of the beam is approximated by

$$t = \delta + \frac{x^2}{2R}. \tag{3}$$

Hence, Eq. (2) may be written in the form

$$\frac{dM}{dx} = -\frac{1}{2} N \frac{x}{R}. \tag{4}$$

Because dt/dx is small in the neighbourhood of $x = 0$, we may apply the beam formula

$$M = EI \frac{d^2w}{dx^2} = \frac{1}{12} Eht^3 \frac{d^2w}{dx^2}, \tag{5}$$

where w is the deflection (positive upwards), h is the strip thickness and E is Young's modulus. Equation (4) now reads

$$\frac{1}{12} Eh \frac{d}{dx}\left(t^3 \frac{d^2w}{dx^2}\right) = -\frac{1}{2} N \frac{x}{R}. \tag{6}$$

The general solution of (6) contains three integration constants. Two boundary conditions are supplied by the requirement that w and its derivatives tend to zero for $x \to \infty$ because the deformation is negligible for large values of x/δ. The third boundary condition is that the slope dw/dx at the minimum cross-section $x = 0$ is zero. Trying a power series solution in t^{-1} it is immediately seen that

$$w = -\frac{2NR}{Eh(\delta + x^2/2R)} \tag{7}$$

satisfies both the differential equation and the boundary conditions.

The bending stress at the inner edge of the beam is given by

$$\sigma_b = \frac{1}{2} Et \frac{d^2w}{dx^2}. \tag{8}$$

Putting

$$\frac{x}{(R\delta)^{1/2}} = \xi, \tag{9}$$

we obtain for the sum of bending stress and tensile stress (due to the tensile force N) at the inner edge and outer edge respectively

$$\sigma_i = \frac{N}{h\delta}\frac{2-\xi^2}{(1+\xi^2/2)^2}, \qquad \sigma_0 = \frac{N}{h\delta}\frac{2\xi^2}{(1+\xi^2/2)^2}. \tag{10}$$

The maximum stress occurs at the inner edge at $x = 0$; its value is $2N/h\delta$, and the stress concentration factor $K = 2$.

3. Cylindrical bar with a large spherical hole. Figure 1 also depicts a meridional section of a cylindrical bar with a *large spherical hole* ($\delta \ll R$), whereas Fig. 2 may be regarded again as an enlarged picture of the upper part to the right of the minimum section. This part is now considered as a cylindrical shell of variable thickness t, given by (3). An element of this shell of circumferential angle $d\varphi$ is drawn in Fig. 4, where the bending moment m and the tensile and shear forces n and q are defined per unit circumferential angle. Obviously the tensile force n is constant. Equilibrium of moments

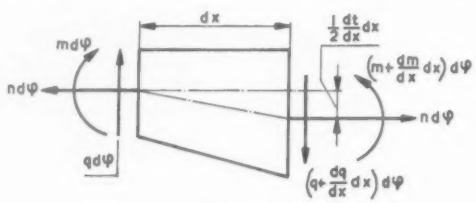

FIG. 4

requires

$$\frac{dm}{dx} = q - \frac{1}{2}n\frac{dt}{dx}, \tag{11}$$

whereas the equilibrium of radial forces is expressed by

$$\frac{dq}{dx} = -t\sigma_\varphi, \tag{12}$$

where σ_φ is the tangential stress in the middle surface. From Hooke's law we have

$$\epsilon_\varphi = \frac{w}{R} = \frac{1}{E}\left(\sigma_\varphi - \nu\frac{n}{Rt}\right), \tag{13}$$

and we obtain

$$\sigma_\varphi = E\frac{w}{R} + \nu\frac{n}{Rt}. \tag{14}$$

Finally the bending moment m is given by the shell formula [4]

$$m = \frac{1}{12}\frac{Et^3R}{1-\nu^2}\frac{d^2w}{dx^2}. \tag{15}$$

Eliminating m, q and σ_φ from (11), (12), (14) and (15) we obtain the differential equation[1]

$$\frac{E}{12(1-\nu^2)}\frac{d^2}{dx^2}\left(t^3\frac{d^2w}{dx^2}\right)+\frac{Et}{R^2}w=-\left(\frac{1}{2}+\nu\right)\frac{n}{R^2}. \tag{16}$$

The boundary conditions are here again that w and its derivatives tend to zero for $x \to \infty$, whereas at the minimum cross-section

$$x=0: \qquad \frac{dw}{dx}=0, \qquad q=0. \tag{17}$$

Trying again a power series solution in t^{-1}, we obtain the simple solution

$$w=-\frac{2(1-\nu^2)(1+2\nu)}{5-4\nu^2}\frac{n}{E}\frac{1}{\delta+x^2/2R}, \tag{18}$$

which satisfies both the differential equation and the boundary conditions.

The bending stress at the inner surface of our shell is given by

$$\sigma_h=\frac{1}{2}\frac{E}{1-\nu^2}t\frac{d^2w}{dx^2}=\frac{n}{\delta R}\frac{1+2\nu}{5-4\nu^2}\frac{1-\frac{3}{2}\xi^2}{[1+(\xi^2/2)]^2}, \tag{19}$$

where ξ is defined by (9). The maximum stress occurs at the inner surface in the minimum section

$$\sigma_{\max}=\frac{n}{\delta R}\left[1+\frac{1+2\nu}{5-4\nu^2}\right]; \tag{20}$$

the nominal stress is given by $n/\delta R$, and the stress concentration factor is therefore

$$K=\frac{(6-4\nu)(1+\nu)}{5-4\nu^2}, \tag{21}$$

varying from 1.2 for $\nu=0$ to 1.5 for $\nu=0.5$.

4. Alternative approach. The stress distribution depends on the two coordinates x and z in Fig. 2. Putting

$$x=(R\delta)^{1/2}\xi, \qquad z=-\delta\eta, \tag{22}$$

we may rewrite the basic equations and boundary conditions (the generalized plane stress equations for the strip problem, and the axially symmetric equations for the cylindrical bar problem) in terms of the new independent variables ξ and η. Developing these equations with respect to the small parameter $\epsilon=\delta/R$, and assuming a similar development for the stresses, a development for the stress concentration factor may be obtained with respect to ϵ. It may well be that such a development does not converge for any finite value of ϵ but even then it will represent an *asymptotic expansion*. The first term has already been obtained by this method for the strip problem, with the result

$$K=2+0(\epsilon), \tag{23}$$

confirming our elementary result of Sec. 2 for $\epsilon \to 0$. Further terms in the strip problem and the similar development for the cylindrical bar problem are now being investigated.

[1]The author is indebted to his collaborator Mr. J. G. Lekkerkerker who pointed out an error in the original analysis.

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THE ANISOTROPIC TENSORS*

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1. Introduction. We consider an nth order Cartesian tensor with components $a_{i_1 i_2 \cdots i_n}$ ($i_p = 1, 2, 3$) in the rectangular Cartesian coordinate system x_i ($i = 1, 2, 3$) and components $a^*_{i_1 i_2 \cdots i_n}$ in the rectangular Cartesian coordinate system x^*_i, where x_i and x^*_i are related by the orthogonal transformation

$$x^*_i = s_{ij} x_j \ , \tag{1.1}$$

with

$$s_{ij} s_{ik} = \delta_{jk} \ , \tag{1.2}$$

where δ_{jk} is the Kronecker delta.

Then,

$$a^*_{i_1 i_2 \cdots i_n} = s_{i_1 j_1} s_{i_2 j_2} \cdots s_{i_n j_n} a_{j_1 j_2 \cdots j_n} \ . \tag{1.3}$$

If

$$a^*_{i_1 i_2 \cdots i_n} = a_{i_1 i_2 \cdots i_n} \ , \tag{1.4}$$

for all s_{ij} satisfying (1.2), then $a_{i_1 i_2 \cdots i_n}$ are said to be the components of an isotropic tensor. If the relation (1.4) is valid only for a subgroup $\{\mathbf{T}\}$ of the group of transformations defined by (1.1) and (1.2) we shall describe $a_{i_1 i_2 \cdots i_n}$ as an *anisotropic tensor*. $a_{i_1 i_2 \cdots i_n}$ may then be described as *invariant under the group of transformations*. The isotropic tensor is, of course, invariant under the orthogonal group.

If (1.3) and (1.4) are valid for a transformation s_{ij}, then it follows that

$$a_{i_1 i_2 \cdots i_n} = s_{i_1 j_1} s_{i_2 j_2} \cdots s_{i_n j_n} a_{j_1 j_2 \cdots j_n} \ . \tag{1.5}$$

If s_{ij} satisfies the relation (1.2), then we readily see from (1.5) that

$$\begin{aligned} a_{i_1 i_2 \cdots i_n} &= s_{i_1 j_1} s_{i_2 j_2} \cdots s_{i_n j_n} a_{j_1 j_2 \cdots j_n} \\ &= s_{j_1 i_1} s_{j_2 i_2} \cdots s_{j_n i_n} a_{j_1 j_2 \cdots j_n} \ . \end{aligned} \tag{1.6}$$

It is shown in Sec. 3 that any tensor which is invariant under the group of transformations $\{\mathbf{T}\}$ may be expressed as the sum of a number of terms, formed from the outer products of a finite set of tensors, with scalar† coefficients. This finite set of tensors,

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†Throughout this paper, we shall employ the term scalar in the following sense. If corresponding quantities φ and φ^* are defined in the coordinate system x_i and in each coordinate system x^*_i, into which x_i is transformed by the group of transformations $\{\mathbf{T}\}$, then if $\varphi^* = \varphi$, we shall say that φ is scalar with respect to the transformation group $\{\mathbf{T}\}$ or, more briefly, scalar.

each of which is invariant under the group of transformations $\{\mathbf{T}\}$, is described as a *tensor basis* for the group of transformations $\{\mathbf{T}\}$.

It is also shown in Sec. 3 how such a tensor basis can be found for any group of transformations $\{\mathbf{T}\}$ for which we can determine a polynomial basis for polynomials in the components of an arbitrary number of vectors, which are form-invariant under the group $\{\mathbf{T}\}$. In Secs. 4, 5 and 6 examples of such tensor bases are obtained for the monoclinic-domatic and rhombic-pyramidal classes of crystal symmetry and for the case of transverse isotropy.

The results in this paper are obtained for three-dimensional space. However, it is immediately evident that similar methods can be used to determine the isotropic or anisotropic tensors for a space of arbitrary dimensions.

2. General considerations. Let P be any scalar polynomial in the components of n vectors $u_i^{(1)}, u_i^{(2)}, \cdots, u_i^{(n)}$, which is form-invariant under a sub-group $\{\mathbf{T}\}$ of the group of orthogonal transformations defined by (1.1) and (1.2). Then, if

$$u_i^{*(r)} = t_{ij}u_j^{(r)} \qquad (r = 1, 2, \cdots, n), \tag{2.1}$$

where t_{ij} is any transformation of $\{\mathbf{T}\}$, we have

$$P(u_i^{(1)}, u_i^{(2)}, \cdots, u_i^{(n)}) = P(u_i^{*(1)}, u_i^{*(2)}, \cdots, u_i^{*(n)}). \tag{2.2}$$

There exists a finite polynomial basis for the polynomials P. Let $I_1, I_2, \cdots, I_N$ denote this polynomial basis. Then,

$$P(u_i^{(1)}, u_i^{(2)}, \cdots, u_i^{(n)}) = Q(I_1, I_2, \cdots, I_N), \tag{2.3}$$

where Q is a polynomial in the indicated variables.

Let us consider the scalar polynomial

$$P = a_{i_1 i_2 \cdots i_n} u_{i_1}^{(1)} u_{i_2}^{(2)} \cdots u_{i_n}^{(n)} \tag{2.4}$$

in the components of the n vectors $u_i^{(1)}, u_i^{(2)}, \cdots, u_i^{(n)}$, the coefficients $a_{i_1 i_2 \cdots i_n}$ in which satisfy the relation

$$\begin{aligned} a_{i_1 i_2 \cdots i_n} &= t_{i_1 j_1} t_{i_2 j_2} \cdots t_{i_n j_n} a_{j_1 j_2 \cdots j_n} \\ &= t_{j_1 i_1} t_{j_2 i_2} \cdots t_{j_n i_n} a_{j_1 j_2 \cdots j_n} \end{aligned} \tag{2.5}$$

for every transformation t_{ij} of the group $\{\mathbf{T}\}$.

It is readily seen that P is form-invariant under the transformations of the group $\{\mathbf{T}\}$. From (2.1), we have

$$a_{i_1 i_2 \cdots i_n} u_{i_1}^{*(1)} u_{i_2}^{*(2)} \cdots u_{i_n}^{*(n)} = a_{i_1 i_2 \cdots i_n} t_{i_1 j_1} t_{i_2 j_2} \cdots t_{i_n j_n} u_{j_1}^{(1)} u_{j_2}^{(2)} \cdots u_{j_n}^{(n)}. \tag{2.6}$$

From (2.6) and (2.5), we obtain the relation

$$a_{i_1 i_2 \cdots i_n} u_{i_1}^{*(1)} u_{i_2}^{*(2)} \cdots u_{i_n}^{*(n)} = a_{i_1 i_2 \cdots i_n} u_{i_1}^{(1)} u_{i_2}^{(2)} \cdots u_{i_n}^{(n)}, \tag{2.7}$$

stating the form-invariance of P.

Since they satisfy the relations (2.5), the 3^n quantities $a_{i_1 i_2 \cdots i_n}$ may be regarded as the components of a tensor, invariant under the group of transformations $\{\mathbf{T}\}$, in the coordinate system x_i and in the coordinate systems into which x_i is transformed by the transformations of the group $\{\mathbf{T}\}$. We thus obtain the result: a scalar polynomial in the components of n vectors, linear in each of the vectors, in which the coefficients are

tensors invariant under the transformations of the group $\{\mathbf{T}\}$, is form-invariant under the transformations of the group $\{\mathbf{T}\}$.

This conclusion can be readily generalized to yield the result that any scalar polynomial in the components of any number of tensors of any rank, in which the coefficients are tensors invariant under the transformation group $\{\mathbf{T}\}$, is form-invariant under the transformations of the group $\{\mathbf{T}\}$.

Now let us consider a polynomial P in the components of n vectors $u_i^{(1)}$, $u_i^{(2)}$, $\cdots$, $u_i^{(n)}$, of the form

$$P = b_{i_1 i_2 \cdots i_n} u_{i_1}^{(1)} u_{i_2}^{(2)} \cdots u_{i_n}^{(n)}, \tag{2.8}$$

which is form-invariant under the group of transformations $\{\mathbf{T}\}$. Then,

$$b_{i_1 i_2 \cdots i_n} u_{i_1}^{(1)} u_{i_2}^{(2)} \cdots u_{i_n}^{(n)} = b_{i_1 i_2 \cdots i_n} u_{i_1}^{*(1)} u_{i_2}^{*(2)} \cdots u_{i_n}^{*(n)}. \tag{2.9}$$

Employing the relation (2.1) in (2.9), we obtain

$$b_{i_1 i_2 \cdots i_n} u_{i_1}^{(1)} u_{i_2}^{(2)} \cdots u_{i_n}^{(n)} = b_{i_1 i_2 \cdots i_n} t_{i_1 j_1} t_{i_2 j_2} \cdots t_{i_n j_n} u_{j_1}^{(1)} u_{j_2}^{(2)} \cdots u_{j_n}^{(n)}. \tag{2.10}$$

Whence,

$$b_{i_1 i_2 \cdots i_n} = b_{i_1 i_2 \cdots i_n} t_{i_1 j_1} t_{i_2 j_2} \cdots t_{i_n j_n}. \tag{2.11}$$

We thus see that the coefficients in a homogeneous scalar polynomial of degree n in the components of n vectors, which is linear in each of the vectors and form-invariant under the transformations of the group $\{\mathbf{T}\}$, are the components of a tensor of rank n, invariant under the group of transformations $\{\mathbf{T}\}$.

This conclusion can be readily generalized to yield the result that any scalar polynomial in the components of any number of tensors of any rank, which is form-invariant under the transformations of the group $\{\mathbf{T}\}$, has coefficients which are tensors, invariant under the group of transformations $\{\mathbf{T}\}$.

3. The tensor basis. Since the polynomial P defined by (2.8) is form-invariant under the group of transformations $\{\mathbf{T}\}$, it must be expressible as a polynomial in the elements of a polynomial basis for polynomials in the elements of n vectors which are form-invariant under $\{\mathbf{T}\}$. Also, since P is linear in each of these vectors, the elements of the polynomial basis which are non-linear in any of the vectors need not be considered. Let J_1, J_2, $\cdots$, J_R be the elements of the polynomial basis which are linear or of degree zero in each of the vectors. Let us suppose that

$$P = \Sigma A_{pq \cdots t} J_p J_q \cdots J_t. \tag{3.1}$$

Then we readily see that

$$\begin{aligned} b_{i_1 i_2 \cdots i_n} &= \frac{\partial^n P}{\partial u_{i_1}^{(1)} \partial u_{i_2}^{(2)} \cdots \partial u_{i_n}^{(n)}} \\ &= \Sigma A_{pq \cdots t} \frac{\partial^n (J_p J_q \cdots J_t)}{\partial u_{i_1}^{(1)} \partial u_{i_2}^{(2)} \cdots \partial u_{i_n}^{(n)}}. \end{aligned} \tag{3.2}$$

The product $J_p J_q \cdots J_t$ occurring in each of the terms under the summation sign in (3.1) is of degree n in the elements of the n vectors $u_i^{(1)}$, $u_i^{(2)}$, $\cdots$, $u_i^{(n)}$ and is linear in each of the vectors. No two of the factors J_p, J_q, $\cdots$, J_t occurring in a single product involve elements of the same vector. Let us assume that J_p is formed from the vectors $u_i^{(1)}$, $u_i^{(2)}$, $\cdots$, $u_i^{(p)}$ only, that J_q is formed from the vectors $u_i^{(p+1)}$, $u_i^{(p+2)}$, $\cdots$, $u_i^{(q)}$ only and J_t is formed from the vectors $u_i^{(s+1)}$, $u_i^{(s+2)}$, $\cdots$, $u_i^{(t)}$ only. Then Eq. (3.2)

may be re-written as

$$b_{i_1 i_2 \cdots i_n} = \Sigma A_{pq\cdots t} \frac{\partial^p J_p}{\partial u_{i_1}^{(1)} \partial u_{i_2}^{(2)} \cdots \partial u_{i_p}^{(p)}} \frac{\partial^{q-p} J_q}{\partial u_{i_{p+1}}^{(p+1)} \partial u_{i_{p+2}}^{(p+2)} \cdots \partial u_{i_q}^{(q)}} \cdots \frac{\partial^{t-s} J_t}{\partial u_{i_{s+1}}^{(s+1)} \partial u_{i_{s+2}}^{(s+2)} \cdots \partial u_{i_t}^{(t)}} \tag{3.3}$$

Since the quantities $A_{pq\cdots t}$ must be scalars under the transformations $\{\mathbf{T}\}$, we see that $b_{i_1 i_2 \cdots i_n}$ must be given as the sum of terms formed from the outer products of a number of tensors of the type $\partial^p J_p / \partial u_{i_1}^{(1)} \partial u_{i_2}^{(2)} \cdots \partial u_{i_p}^{(p)}$, obtained by differentiating the elements $J_1, J_2, \cdots, J_R$ of the polynomial basis with respect to the vectors from which they are formed. The coefficients of these terms are scalars with respect to the transformation group $\{\mathbf{T}\}$.

4. The monoclinic system—domatic class. Monoclinic symmetry may be described with relation to three preferred directions in space. We denote these directions by the unit vectors $\mathbf{h}_1$, $\mathbf{h}_2$ and $\mathbf{h}_3$. The vectors $\mathbf{h}_2$ and $\mathbf{h}_3$ are not at right-angles and the vector $\mathbf{h}_1$ is perpendicular to the plane defined by $\mathbf{h}_2$ and $\mathbf{h}_3$.

Let us choose as reference frame a rectangular Cartesian coordinate system x_i, the x_1 axis of which coincides in direction with the vector $\mathbf{h}_1$. The axes x_2 and x_3 may be in arbitrary perpendicular directions in the $\mathbf{h}_2\mathbf{h}_3$ plane. Then, the group of transformations $\{\mathbf{T}\}$ associated with the monoclinic-domatic symmetry class consists of the identity transformation $\mathbf{T}_1$ and the transformation $\mathbf{T}_2$ defined by

$$\mathbf{T}_2 = \begin{bmatrix} -1, & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & 1 \end{bmatrix}. \tag{4.1}$$

Let us consider n vectors with components $u_i^{(1)}, u_i^{(2)}, \cdots, u_i^{(n)}$ in the coordinate system x_i. Any polynomial

$$P(u_1^{(1)}, u_2^{(1)}, u_3^{(1)}, u_1^{(2)}, u_2^{(2)}, u_3^{(2)}, \cdots, u_1^{(n)}, u_2^{(n)}, u_3^{(n)})$$

is unaltered by the transformation $\mathbf{T}_1$ and is transformed into

$$P(-u_1^{(1)}, u_2^{(1)}, u_3^{(1)}, -u_1^{(2)}, u_2^{(2)}, u_3^{(2)}, \cdots, -u_1^{(n)}, u_2^{(n)}, u_3^{(n)})$$

by the transformation $\mathbf{T}_2$. The necessary and sufficient condition that P be form-invariant under the transformation $\mathbf{T}_2$ is that it be expressible as a polynomial in

$$u_2^{(r)}, u_3^{(r)}, u_1^{(r)} u_1^{(s)} \qquad (r, s = 1, 2, \cdots, n). \tag{4.2}$$

This result follows immediately from the first main theorem of classical invariant theory†, according to which a polynomial basis for polynomials in ξ_r, η_r $(r = 1, 2, \cdots, n)$, which are form-invariant under interchange of ξ_r and η_r, is given by $\xi_r + \eta_r$ and $\xi_r \eta_s + \xi_s \eta_r$. Taking

$$(\xi_1, \xi_2, \cdots, \xi_n) = (u_1^{(1)}, u_2^{(1)}, u_3^{(1)}, \cdots, u_1^{(n)}, u_2^{(n)}, u_3^{(n)})$$

†See, for example, H. Weyl, *The classical groups, their invariants and representations*, Princeton Univ. Press, Princeton, N. J., 1946, p. 36 et seq.

and

$$(\eta_1 , \eta_2 , \cdots , \eta_n) = (-u_1^{(1)}, u_2^{(1)}, u_3^{(1)}, \cdots , -u_1^{(n)}, u_2^{(n)}, u_3^{(n)})$$

and neglecting redundant elements, we obtain the result (4.2).

The expressions (4.2) form a polynomial basis for polynomials in the components of the n vectors $u_i^{(r)}$ $(r = 1, 2, \cdots , n)$ which are form-invariant under the transformations characterizing the symmetry class considered. From the results of the previous section, we see that the quantities $\partial u_2^{(r)}/\partial u_i^{(r)}$, $\partial u_3^{(r)}/\partial u_i^{(r)}$ and $\partial^2 (u_1^{(r)} u_1^{(s)})/\partial u_i^{(r)} \partial u_i^{(s)}$ $(r \neq s)$ define a basic set of anisotropic tensors for the monoclinic-domatic class in terms of which an arbitrary anisotropic tensor for this class can be expressed. We note that $\partial u_2^{(r)}/\partial u_i^{(r)}$ defines the vector with components

$$\delta_{2i} = (0, 1, 0) = (\alpha_i) \quad \text{(say)} \tag{4.3}$$

in the coordinate system x_i , that $\partial u_3^{(r)}/\partial u_i^{(r)}$ defines the vector with components

$$\delta_{3i} = (0, 0, 1) = (\beta_i) \quad \text{(say)} \tag{4.4}$$

in the coordinate system x_i and $\partial^2 (u_1^{(r)} u_1^{(s)})/\partial u_i^{(r)} \partial u_i^{(s)}$ $(r \neq s)$ defines the tensor of rank 2 with components

$$\begin{bmatrix} 1, & 0, & 0 \\ 0, & 0, & 0 \\ 0, & 0, & 0 \end{bmatrix} = (\gamma_{ij}) \quad \text{(say)} \tag{4.5}$$

in the coordinate system x_i . Further, any tensor with components $a_{ij\ldots k}$ in the coordinate system x_i , which is invariant under the transformations of the monoclinic-domatic class, is expressible as the sum of terms formed from outer products of α_i , β_i and γ_{ij} with scalar coefficients which are invariant under the transformations of the class.

5. The rhombic crystal system—rhombic-pyramidal class. Rhombic symmetry may be described with relation to three preferred directions in space, defined by the mutually perpendicular unit vectors $\mathbf{h}_1$, $\mathbf{h}_2$, $\mathbf{h}_3$. We choose as our reference system the rectangular Cartesian coordinate system x_i , the axes of which coincide in direction with the vectors $\mathbf{h}_i$. The group of transformations $\{\mathbf{T}\}$ associated with the rhombic-pyramidal class is composed of the identity transformation $\mathbf{T}_1$ and the transformations $\mathbf{T}_3$, $\mathbf{T}_4$ and $\mathbf{T}_5$ defined by

$$\mathbf{T}_3 = \begin{bmatrix} 1, & 0, & 0 \\ 0, & -1, & 0 \\ 0, & 0, & 1 \end{bmatrix}, \quad \mathbf{T}_4 = \begin{bmatrix} 1, & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{T}_5 = \begin{bmatrix} 1, & 0, & 0 \\ 0, & -1, & 0 \\ 0, & 0, & -1 \end{bmatrix}. \tag{5.1}$$

We can determine, in a manner similar to that adopted in Sec. 4, a polynomial basis for polynomials P in the components of n vectors which are form-invariant under the transformations (5.1). If these vectors have components $u_i^{(1)}$, $u_i^{(2)}$, $\cdots$, $u_i^{(n)}$ in the coordinate system x_i , then invariance under the transformation $\mathbf{T}_3$ implies that P must be expressible as a polynomial P' (say) in the quantities $u_1^{(r)}$, $u_3^{(r)}$, $u_2^{(r)} u_2^{(s)}$ $(r, s = 1, 2, \cdots , n)$. Thus,

$$P = P'(u_1^{(r)}, u_3^{(r)}, u_2^{(r)} u_2^{(s)}). \tag{5.2}$$

Invariance of P' under the transformation $\mathbf{T}_4$ imposes on P' the limitation

$$P'(u_1^{(r)}, u_3^{(r)}, u_2^{(r)}u_2^{(s)}) = P'(u_1^{(r)}, -u_3^{(r)}, u_2^{(r)}u_2^{(s)}). \tag{5.3}$$

It is readily seen that P' must be expressible as a polynomial in the quantities $u_1^{(r)}$, $u_2^{(r)}\,u_2^{(s)}$ and $u_3^{(r)}\,u_3^{(s)}$. Such a polynomial is obviously invariant under the transformation $\mathbf{T}_5$ and hence under all the transformations of the group $\{\mathbf{T}\}$.

Following the procedure described in Sec. 3, we see that $\partial u_1^{(r)}/\partial u_i^{(r)}$ $(= \alpha_i$, say), $\partial^2(u_2^{(r)}\,u_2^{(s)})/\partial u_i^{(r)}\,\partial u_j^{(s)}$ $(r \neq s)$ $(= \beta_{ij}$, say), $\partial^2(u_3^{(r)}\,u_3^{(s)})/\partial u_i^{(r)}\,\partial u_j^{(s)}$ $(r \neq s)$ $(= \gamma_{ij}$, say) define a basic set of anisotropic tensors for the rhombic-pyramidal class. We have

$$(\alpha_i) = (\delta_{1i}) = (1, 0, 0)$$

$$(\beta_{ij}) = \begin{bmatrix} 0, & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & 0 \end{bmatrix} \quad \text{and} \quad (\gamma_{ij}) = \begin{bmatrix} 0, & 0, & 0 \\ 0, & 0, & 0 \\ 0, & 0, & 1 \end{bmatrix}. \tag{5.4}$$

6. Transverse isotropy. Transverse isotropy may be described with relation to a single preferred direction in space defined by a unit vector $\mathbf{h}$. We choose as our reference frame a rectangular Cartesian coordinate system x_i , the x_3 axis of which coincides in direction with the vector $\mathbf{h}$. Then, the group of transformations $\{\mathbf{T}\}$ associated with transverse isotropy with respect to the direction of $\mathbf{h}$ is composed of the transformation $\mathbf{T}_2$ defined by (4.1), the transformation $\mathbf{T}_\omega$ defined by

$$\mathbf{T}_\omega = \begin{bmatrix} \cos\omega, & \sin\omega, & 0 \\ -\sin\omega, & \cos\omega, & 0 \\ 0 & , & 0 & , & 1 \end{bmatrix} \qquad (0 \leq \omega < 2\pi), \tag{6.1}$$

and the transformation $\mathbf{T}_2\mathbf{T}_\omega$.

We can determine a polynomial basis for polynomials P in the components of n vectors which are linear in each of the vectors and form-invariant under the transformations $\mathbf{T}_2$, $\mathbf{T}_\omega$ and $\mathbf{T}_2\mathbf{T}_\omega$. Let $u_i^{(1)}, u_i^{(2)}, \cdots, u_i^{(n)}$ be the components of the n vectors in the coordinate system x_i . Then, the limitations imposed on P by the condition that it be form-invariant under the transformations $\mathbf{T}_2$, $\mathbf{T}_\omega$ and $\mathbf{T}_2\mathbf{T}_\omega$ are

$$\begin{aligned} P(u_1^{(r)}, u_2^{(r)}, u_3^{(r)}) &= P(u_1^{(r)}\cos\omega + u_2^{(r)}\sin\omega, -u_1^{(r)}\sin\omega + u_2^{(r)}\cos\omega, u_3^{(r)}) \\ &= P(-u_1^{(r)}, u_2^{(r)}, u_3^{(r)}). \end{aligned} \tag{6.2}$$

It is seen that the dependence of P on $u_3^{(r)}$ gives rise to no restrictions on the form of P and, therefore, a polynomial basis for P is formed by $u_3^{(r)}$ $(r = 1, 2, \cdots, n)$ and the polynomial basis for polynomials P' which satisfy the conditions

$$\begin{aligned} P'(u_1^{(r)}, u_2^{(r)}) &= P'(u_1^{(r)}\cos\omega + u_2^{(r)}\sin\omega, -u_1^{(r)}\sin\omega + u_2^{(r)}\cos\omega) \\ &= P'(-u_1^{(r)}, u_2^{(r)}), \end{aligned} \tag{6.3}$$

and are linear in the two-dimensional vectors $u_\alpha^{(r)}$ $(\alpha = 1, 2)$. The relations (6.3) imply that P' is form-invariant under the group of two-dimensional orthogonal transformations, proper and improper. Now, the quantities $u_\alpha^{(r)}\,u_\alpha^{(s)}$ $(r, s = 1, 2, \cdots, n; \alpha = 1, 2)$ form a polynomial basis for polynomials in the components of n two-dimensional vectors $u_\alpha^{(r)}$

which are form-invariant under the full orthogonal group†. Since P' is linear in the vectors, we can omit those elements $u_\alpha^{(r)}\, u_\alpha^{(s)}$ for which $r = s$. We thus have the result that the quantities

$$u_3^{(r)},\ u_\alpha^{(r)} u_\alpha^{(s)} \qquad (r, s = 1, 2, \cdots, n; r \neq s) \tag{6.4}$$

form a polynomial basis for polynomials in the components of n vectors $u_i^{(r)}$ which are linear in each of the vectors and form-invariant under the transformations associated with transverse isotropy about the x_3 axis.

As in the previous sections, we immediately see that a basic set of anisotropic tensors for transverse isotropy about the x_3 axis is defined by $\partial u_3^{(r)}/\partial u_i^{(r)}$ $(= \alpha_i$, say) and $\partial(u_\alpha^{(r)}\, u_\alpha^{(s)}/\partial u_i^{(r)}\, \partial u_j^{(s)}$ $(= \beta_{ij})$ $(r \neq s; \alpha = 1, 2; i, j = 1, 2, 3)$. We obtain immediately

$$(\alpha_i) = (\delta_{3i}) = (0, 0, 1)$$

and

$$(\beta_{ij}) = (\delta_{1i}\delta_{1j} + \delta_{2i}\delta_{2j}) = \begin{bmatrix} 1, & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & 0 \end{bmatrix}. \tag{6.5}$$

ON A PRINCIPLE OF RECIPROCITY BETWEEN HIGH- AND LOW-FREQUENCY PROBLEMS CONCERNING LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER*

By AUREL WINTNER (*The Johns Hopkins University*)

1. The following considerations, in which nomenclature and point of view will be the same as in two earlier notes[1], deal with the oscillatory case of the differential equation

$$x'' + f(t)x = 0, \tag{1}$$

in which $f(t)$ is a real-valued, continuous function for large positive t, and $x(t)$ is any real-valued solution distinct from the trivial solution $x(t) \equiv 0$.

If (1) is oscillatory when $f(t) = f_1(t)$, and if $f_1(t) \leqq f_2(t)$, then (1) is oscillatory when $f(t) = f_2(t)$. This follows from Sturm's comparison theorem, which implies also the following fact: If (1) is oscillatory and if $d_n = d_n(f) = d_n(f; x)$ denotes the distance $t_{n+1} - t_n$, where $t_n = t_n(f; x)$ is the nth zero of a solution $x = x(t) = x_f(t)$ of (1), then $\limsup d_n(f_1)/d_n(f_2) < 1$, where $n \to \infty$, holds whenever $f_2(t) - f_1(t)$ exceeds a positive lower bound as $t \to \infty$.

By choosing $f_1(t) = \omega^2$, where ω is a positive constant, and letting $\omega \to \infty$ or $\omega \to 0$, it follows that $d_n(f) \to 0$ as $n \to \infty$ if

$$f(t) \to \infty \quad \text{as} \quad t \to \infty, \tag{2}$$

†See, for example, H. Weyl, loc. cit. p. 52 et. seq.

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[1]A. Wintner, Quart. Appl. Math. **7**, 115-117 (1949) and **13**, 192-195 (1955). These two papers will be referred to as [1] and [2] respectively.

and that $d_n(f) \to \infty$ as $n \to \infty$ if

$$f(t) \to 0 \quad \text{as} \quad t \to \infty \tag{3}$$

(provided that (1) is oscillatory; this additional proviso is necessary, even if

$$f(t) > 0, \tag{4}$$

in the case (3), but is automatically satisfied in the case (2), or even if just $f(t) \geqq \omega^2$ holds for some positive constant.) For this reason, (1) will be called of high frequency in the (oscillatory subcase of the) case (3).

2. If $f(t) = \lambda/t^2$ (for $1 \leqq t < \infty$), where λ is a positive constant, then (4) and (3) are satisfied (and

$$\int^{\infty} f(t)\, dt = \infty \tag{5}$$

fails to hold). But it depends on the numerical value of λ (hence, not merely on the order of magnitude of $f(t) > 0$ for large t) whether (1) is or is not oscillatory in this case. In fact, the remark of A. Kneser, quoted in[2] [2], shows that the case $f(t) = \lambda/t^2$ of (1) is oscillatory only if $\lambda > 1/4$, rather than just $\lambda > 0$, is assumed.

Thus (3) and the negation of (5) together are compatible with the oscillatory as well as with the non-oscillatory character of (1) in the case (4). On the other hand, (1) must be oscillatory if (4) and (5) hold; in fact, more than this was proved in [1].

In what follows, *both* (4) *and* (5) *will be assumed*. This will imply, in particular, that (1) is oscillatory even if (3) holds.

3. There is an extensive literature[3] dealing with the asymptotic behavior (as $t \to \infty$) of the solutions $x(t)$ of (1) in the case (2) of high frequency. Let any such theorem be referred to as of type (H), and let (L) refer to theorems dealing with the asymptotic behavior of the solutions $x(t)$ of (1) in the case (3) of low frequency.

An inspection of the literature reveals a certain empirical dualism between these two types of theorems: As soon as a theorem of type (H) is discovered, its wording and proof can usually be altered so as to supply a theorem of type (L), and conversely. But as far as the literature consulted goes, this principle of transfer appears only *ad hoc*, since its applicability is not assured *a priori*.

The following consideration will dispose of this anomaly, by exhibiting a formal fact which underlies the principle of tranfer, $(H) \to (L)$ or $(L) \to (H)$.

4. It will turn out that what actually matters in this regard is neither (2) nor (3) but a connection between the differential equations (1) and

$$\ddot{x} + g(s)x = 0, \tag{6}$$

where the dots, in contrast to the primes of (1), refer to differentiations with respect to the independent variable $s = s(t)$ defined by

$$ds = f(t)\, dt, \tag{7}$$

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[2]Concerning the brackets see the preceding footnote.

[3]cf. G. Sansone, *Equazioni differenziali nel campo reale*, vol. 2 (2nd ed., 1949), chap. VII; and F. G. Tricomi, *Equazioni differenziali* (2nd ed., 1953), chap. III-IV. These expositions, and the papers mentioned in the first footnote above, contain further references to the literature.

and the coefficient functions of (1) and (6) are reciprocals by virtue of (7) or

$$dt = ds/g(s), \tag{8}$$

that is,

$$f = 1/g \quad \text{or} \quad g = 1/f. \tag{9}$$

Note that, because of (4) and (7), the correspondence $s = s(t)$ between the given t-range and the s-range is a one-to-one correspondence, which is continuously differentiable along with its inverse, $t = t(s)$, and that, by virtue of (4), (7) and (5),

$$t \to \infty \quad \text{if and only if} \quad s \to \infty. \tag{10}$$

It is clear from (9) that (6) is of low frequency or of high frequency according as (1) is of high frequency or of low frequency. But the connection between (1) and (6), being of a formal nature, will be such as to apply also in the intermediary cases, cases in which neither (2) nor (3) is satisfied.

5. The proof depends, as in [1], on an appropriate application of Riccati's equation (an application which, however, cannot be the same as in [1]; cf. Section 6 below).

First, if I is a t-interval on which a solution $x(t)$ of (1) does not vanish, then it is possible to form the logarithmic derivative

$$m(t) = x'(t)/x(t) \tag{11}$$

at every point t of I, and $m = m(t)$ will be a solution of Riccati's differential equation

$$m' + m^2 + f(t) = 0. \tag{12}$$

In fact, if (1) is divided by x, then (12) follows from (11).

Next, suppose that not only $x(t)$ but also $x'(t)$ is distinct from 0 at every point t of I. Then (11) shows that it is possible to form on I the function

$$n(t) = -1/m(t), \tag{13}$$

and it is seen from (13) that (12) is equivalent to

$$n' + 1 + f(t)n^2 = 0. \tag{14}$$

Finally, if (14) is divided by $f(t)$ (which, in view of (4), is allowed), then the definitions, (9) and (7), of s and g show that (14), where $n' = dn/dt$, can be written in the form

$$\dot{n} + n^2 + g(s) = 0, \tag{15}$$

where $\dot{n} = dn/ds$.

This proves that, as a matter of formal coincidence, (1) is equivalent to (6) by virtue of (9) and (7). In fact, (15) belongs to (6) in the same way (that is, as the Riccati resolvent of the linear differential equation) as (12) belongs to (1).

6. Actually, the proof is not yet complete. For, on the one hand, Section 5 confines t to a t-range I on which neither $x(t)$ nor $x'(t)$ has a zero and, on the other hand, the problems of the asymptotic behavior concern themselves with the t-range I which is a half-line $t_0 \leqq t < \infty$ on which $x(t)$ (hence $x'(t)$ as well) has an infinity of zeros, since (1) is oscillatory. In fact, the oscillatory character of (1) is implied by (4) and (5), and (5) cannot be omitted, since otherwise (10) becomes false. This difficulty can however readily be disposed of, as follows:

Instead of a single real-valued solution $x(t)$ of (1), choose for (1) two solutions, $x = x(t)$ and $x = y(t)$, which are real-valued and linearly independent. In view of the superposition principle, the (complex-valued) function

$$z(t) = x(t) + iy(t) \tag{16}$$

is a solution of the (real) differential equation (1). But

$$z(t) \neq 0 \quad \text{and} \quad z'(t) \neq 0 \tag{17}$$

now hold on I when I is the entire half-line under consideration ($t_0 \leqq t < \infty$). In fact, since $x = x(t)$ and $x = y(t)$ are linearly independent solutions of (1), their Wronskian $xy' - yx'$ is a non-vanishing constant; hence neither x and y nor x' and y' can vanish at the same t, and so it is clear from (16) that both inequalities (17) are satisfied at every t.

Owing to the first of the inequalities (17), the logarithmic derivative

$$m(t) = z'(t)/z(t) \tag{18}$$

exists on the entire half-line I. On the other hand, the second of the inequalities (17) assures that (13) defines the function $n(t)$ on the entire half-line I if $m(t)$ is defined by (18).

Finally, since $x = z(t)$ is a solution of (1), Riccati's deduction of (12) from (11) and the case $x = x(t)$ of (1) remains unaltered if (11) and (1) are replaced by (17) and the case $x = z(t)$ of (1) respectively. Since the transition from (12) to (15) had nothing to do with the reality of $m(t)$, the proof is now complete.

ON THE ROTATION OF AN INCOMPRESSIBLE CONTINUOUS MEDIUM IN PLANE MOTION*

By WALTER NOLL (*Carnegie Institute of Technology*)

We consider a plane motion of an incompressible medium with homogeneous density $\rho = \text{const}$. We assume that the plane region occupied by the medium is of the following type: at each instant it is bounded by a finite number of smooth curves. If it is finite all but one of the closed boundary curves move rigidly. If it extends to infinity all finite closed boundary curves move rigidly. We label this motion by I. Now we consider another motion, labeled by II, which is obtained from I by superposing a uniform rotation with angular velocity $\omega = \text{const}$. about a fixed axis perpendicular to the plane of the motion.

It will be shown that motion II is dynamically possible whenever I is, and that the stresses in II differ from those in I only by a scalar stress given explicitly in (10) below.

G. I. Taylor [1] proved this theorem for the special case of the irrotational motion of an inviscid fluid caused by rigid cylindrical bodies moving in an infinite container. It has been shown recently that Taylor's result continues to hold for linear viscous fluids [2] and for a special type of non-linear viscous fluids [4]. The main purpose of this paper is to point out that Taylor's theorem remains valid for any material whatsoever (it may even be an anisotropic solid). The essential assumptions are that the medium be incompressible and that the motion be plane.

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A motion with velocity $\mathbf{v}$ is dynamically possible if there is a stress tensor S such that the continuity equation, the equation of motion, and the constitutive equation of the material are satisfied with $\mathbf{v}$ and S. The first two of these equations are in our case

$$\operatorname{div} \mathbf{v} = 0 \tag{1}$$

$$\operatorname{div} S + \rho \mathbf{f} = \rho \dot{\mathbf{v}}, \tag{2}$$

where $\mathbf{f}$ is the density of the extraneous forces. In general, the constitutive equation will serve to determine the stress S by the motion (and possibly certain initial data). However, in an incompressible medium the motion determines the stress only up to a scalar stress pI (I = unit tensor) so that only the extra stress $T = S + pI$ is completely determined. It follows from the principle of isotropy of space (see [3], p. 19) that the constitutive equation is not altered by the superposition of any rigid rotation.

We describe motion II in a frame of reference which rotates with the superposed rotation. Then both I and II will have the same velocity field $\mathbf{v}_{\mathrm{I}} = \mathbf{v}_{\mathrm{II}} = \mathbf{v}$. Therefore, by the principle just mentioned, the extra stresses must be the same for both motions:

$$T_{\mathrm{I}} = p_{\mathrm{I}} I + S_{\mathrm{I}} = p_{\mathrm{II}} I + S_{\mathrm{II}} . \tag{3}$$

The equation of continuity (1) is the same for both motions. For a plane motion, (1) implies the existence of a stream function ψ such that

$$\mathbf{v} = -\operatorname{curl} (\mathbf{e}\psi) \tag{4}$$

where $\mathbf{e}$ is a unit vector perpendicular to the plane of motion. The assumptions on the region mentioned in the beginning ensure that ψ is single valued. The equation of motion (2) for I is

$$\operatorname{div} S_{\mathrm{I}} + \rho \mathbf{f} = \rho \dot{\mathbf{v}} \tag{5}$$

In the rotating frame, however, we must add to $\mathbf{f}$ a field $\mathbf{g}$ which represents the centrifugal and Coriolis forces, so that we get

$$\operatorname{div} S_{\mathrm{II}} + \rho(\mathbf{f} + \mathbf{g}) = \rho \dot{\mathbf{v}} \tag{6}$$

(It is understood that the extraneous forces $\mathbf{f}$ are to be rotated also). It follows from (3), (5), and (6) that motion II will be dynamically possible if and only if p_{II} can be chosen in such a way that

$$\rho \mathbf{g} = \operatorname{grad} (p_{\mathrm{II}} - p_{\mathrm{I}}) \tag{7}$$

holds, i. e., if and only if $\mathbf{g}$ has a single valued potential. This is not the case for a general three-dimensional motion. But for a plane motion an easy analysis (see [1], p. 101) shows that

$$\mathbf{g} = \operatorname{grad} (2\omega\psi + \tfrac{1}{2}\omega^2 r^2), \tag{8}$$

where r is the distance from the axis of rotation. It follows from (8), (7) and (3) that the pressure p_{II} and the stress S_{II} of motion II are given by

$$p_{\mathrm{II}} = p_{\mathrm{I}} + \rho(2\omega\psi + \tfrac{1}{2}\omega^2 r^2), \tag{9}$$

$$S_{\mathrm{II}} = S_{\mathrm{I}} - \rho(2\omega\psi + \tfrac{1}{2}\omega^2 r^2) I. \tag{10}$$

From (10), the forces exerted on the boundaries in motion II can be obtained in the same way as was done by Taylor [1].

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BOOK REVIEWS

(*Continued from p. 282*)

The theory of functions of real variables. By Lawrence M. Graves. McGraw-Hill Book Co., Inc., New York, Toronto, London, 1956. xii + 375 pp. $7.50.

Published ten years ago, the first edition of this book is now widely known and has been reviewed in detail: see, e.g., Math. Rev., **8** (1947), 319-20, or Bull. Amer. Math. Soc., **54** (1948), 487-9. Except for a few rewritten passages, the second edition differs from the first only in that two new chapters have been appended, and we shall confine comment to these.

Chapter XIII, on sets and transfinite numbers, begins with an amplified account of sets and functions (= relations) from the non-formal "common-sense" point of view adopted by the author in Chapter I. In the same spirit there follows a discussion of order leading up to definitions of ordinal and cardinal number which are essentially von Neumann's: an ordinal is a set which (i) is well-ordered by the class-membership relation and (ii) has each of its members as a subset; and a cardinal is an initial ordinal in a cardinal-similarity class of ordinals. Next, the maximality principle (Zorn's Lemma) is stated and studied in a variety of equivalent forms. The chapter closes with an introduction to ordinal and cardinal algebra. Chapter XIV discusses, for metric spaces, basic facts concerning convergence, completeness, connectedness, separability, compactness, and category. Here the references for further study might well have included such recent items as J. L. Kelley's *General Topology*, 1955, as well as the older works listed. Compared with the rest of the book both of the added chapters are notable for their profusion of well selected exercises.

TRUMAN BOTTS

Automata studies. Edited by C. E. Shannon and J. McCarthy. Princeton University Press, New Jersey, 1956. viii + 285 pp. $4.00.

The recent development of high-speed digital computers and of electronic control equipment has revived interest in questions on the ultimate limits of a "mechanical brain" and on how much of animal or human mental activity can be described in physical terms. The present collection of studies represents some of the contributions of mathematics to these perennially interesting fields of speculation.

The articles on the properties and limitations of computers are based on the work of Turing, in which he showed that the action of most digital computers is equivalent to an appropriate sequence of actions of a machine with simply describable properties, the limitations of which could be studied by known mathematical methods. The articles in this collection presuppose a knowledge of Turing machines and carry further the analysis of their behavior.

The papers more specifically related to the analysis of mental action are concerned with the properties of "automata," machines built up of "neurons" having a simple, all-or-none behavior analogous to living nerve cells. The first paper of the collection, by S. C. Keene, gives a readable introduction to the subject. Some of the later papers discuss limitations and extensions of the basic ideas.

Of course an automaton, built up of infallible "neurons" could perhaps duplicate some of the human brain's abilities in the development of formal logic. To duplicate actual learning and memory we have to introduce the possibility of failure of an element from time to time. The study of automata with unreliable elements is of considerable interest, for it may point the way to methods of improving the reliability of the machine as a whole, whether it be one of our all-too-fallible present day computers or our own brains. The basic article on this subject, in the collection, is by von Neumann. One can hope that others will explore further the ideas he outlines here.

The articles are organized into three sections: the first on Finite Automata, the next on Turing Machines and the last on Synthesis of Automata. The papers are recommended to anyone interested in this general field, though some of them make pretty hard reading. If the interesting Introduction, by John McCarthy and Claude Shannon, had been expanded by a dozen pages or so to provide a review of of previous work and a bibliography of important papers, this volume would have constituted a much-needed review of the field. As it is, the collection is useful and thought-provoking.

PHILIP M. MORSE

Elementary differential equations. By William Ted Martin and Eric Reissner. Addison-Wesley Publishing Co., Inc., Cambridge, Mass., 1956. xi + 260 pp. $5.50.

This book is intended primarily as a text book for advanced undergraduate science and engineering students. The book contains six chapters relating to ordinary differential equations, one chapter on finite difference equations and one chapter on partial differential equations. Emphasis is on methods of solution for which the authors attempt to give some motivation although this is clearly rather difficult at times. There are numerous examples and problems including several examples to illustrate that the method being discussed does not always succeed. These latter examples show to the student better than words could describe that the subject goes much deeper.

GORDON F. NEWELL

Advances in mechanics: Volume IV. Editors: H. L. Dryden, Th. von Kármán. Managing Editor: G. Kuerti. Associated Editors: F. H. Van den Dungen, L. Howarth, J. Pérès. Academic Press, Inc., New York, 1956. x + 413 pp. $10.00.

The present volume is the first to appear since the death of Richard von Mises, the initiator of this series; it contains the following articles: The Turbulent Boundary Layer, by Francis H. Clauser (51 pp.)—Nonlinear Elasticity, by T. C. Doyle and J. L. Ericksen (63 pp.)—Physical and Statistical Aspects of Fatigue, by A. M. Freudenthal and E. J. Gumbel (42 pp.)—Three-Dimensional Boundary Layer Theory, by Franklin K. Moore (70 pp.)—Dislocation Theory of Plasticity of Methals, by G. Schoeck (51 pp.)—The Poincaré-Lighthill-Kuo Method, by H. S. Tsien (69 pp.)—On the Concept of Elastic Stability, by Hans Ziegler (53 pp.).

W. PRAGER

Les Épreuves sur Échantillon. By Maurice Dumas. Centre d'Études Mathematiques en vue des applications, of the Centre National de la Récherche Scientifique, Paris, 1955. 170 pp. $2.90.

This is a short monograph on sampling inspection intended for scientists and engineers who are supposed to have the mathematical background, but not the available time, to read up on the relevant theory. The presentation is thought-provoking, and perhaps a little complicated. Certainly it is far more sophisticated than that of the numerous American books and manuals on practical statistical quality control which have been appearing in profusion lately. Incidentally, the preface is by no less a celebrity than Professor Maurice Fréchet.

The author arranges his book in this way: First there is a summary in a telegraphic style of the main formulas of the book and procedures for applying them. Then there is a careful chapter on the terminology, techniques and rationale of sampling inspection. This chapter includes a brief but clear non-mathematical discussion of the relations between statistics, practical probabilities, and mathematical theory of probability. After this comes a general description, with numerical examples, of the four categories of problems which the author intends to discuss: Estimation, acceptance inspection, comparison of lots with each other and with fixed standards, and statistical quality control in the Shewhart sense.

In the next chapter (we have now arrived at Chapter III), detailed procedures for attacking these four types of problems are given for the case of sampling by attributes. In the last chapter, Chapter IV, methods for the case of sampling by variables are given. The idea of the arrangement is that a reader who wants only to get a general idea of what sampling inspection is all about, can do it by reading only the first part of the book, omitting the last two chapters.

Mathematical notations and formulas are used liberally, but there are no proofs. For these, the reader is referred chiefly to two large works which stand behind this one: a six-hundred and thirty page work on industrial statistical methods written by the author in collaboration with P. Maheu, and a work called *Technique modernes de contrôle des fabrications*, by J. Mothes. There is a paucity of acknowledgments to those whom the author calls "the Anglo-Saxons", although much of the terminology is drawn directly from Fisher, Dodge and Romig, Neyman and Pearson, and Shewhart. (By the way,

Dr. Shewhart's name is apparently never mentioned.) This is certainly excusable in view of the nature of the book, but Dumas indicates in his brief bibliography that he has another reason for not sensing a debt to the Anglo-Saxons too keenly. He says that anyhow, in a 1925 Mémorial de l'Artillerie Française entitled *Sur une interprétation des conditions de recette*, he anticipated the relevant portions of the Dodge-Romig and Neyman-Pearson theories.

The author generally limits himself to strictly classical procedures and formulas. There are short sections on sequential analysis, and some of the tests of Barnard are given, but otherwise this book could have been written at least twenty-five years ago. There is, for example, nothing about non-parametric analysis and nothing about sampling by variables when lot quality is measured by a percentage of non-conforming items. Perhaps the author should read more Anglo-Saxon articles on mathematical statistics.

However there is an interesting emphasis on a posteriori probability, which this reviewer considers to be of importance in sampling inspection from a pragmatic rather than from a philosophical point of view. Almost all sampling inspection is performed on lots which themselves can be considered as probability samples from a larger population, and one way to take this into account is to use Bayes' Theroem. The essentially stochastic nature of inspection lots does not seem to be sufficiently emphasized in the current Anglo-Saxon literature of the subject.

In the case of sampling by variables, the author deals at some length with the technique of plotting observed frequency distributions on normal probability paper. He calls such a plot a "graphique de Henry". In this connection, readers of the monograph under review might do well to look at an article entitled "Use of Normal Probability Paper", by Chernoff and Lieberman, in the Journal of the American Statistical Association, vol. 49, 1954, pp. 778-785, which discusses the question of optimal choices of the ordinates for various problems.

All in all, this book should be quite successful in meeting its aims, especially as regards French readers. In fact, some Anglo-Saxons who meet the specifications of the intended audience, and who can read the rather turgid prose which seems to be in favor nowadays in French scientific circles (many semi-colons and colons) might find the book useful.

J. H. CURTISS

Zur Ermittlung von Unterschallströmungen mit der Transformationsmethode bei quadratischer Approximation der Adiabate. By Hans Schubert and Erich Schincke. Akademie-Verlag, Berlin, 1955. 32 pp. $.55.

This monograph contains a treatment of the problem of finding the compressible inviscid flow past a symmetrical two-dimensional object when the pressure-density relation is approximated by such a function that the equation for the dimensionless stream function ψ becomes $\Delta\psi + k^2\psi = 0$. Here k is a constant and the independent variables are distorted hodograph variables. It is shown that this p-v law agrees with the usual isentropic relation to quadratic terms in their series in powers of the velocity. Solutions are then expected as the sum of a singular term closely related to the singular part of the hodograph solution of the corresponding incompressible problem and a series of non-singular terms whose coefficients are determined by the boundary conditions in the physical plane and which are therefore to be extracted as the solution of a non-linear algebraic problem. The authors note that the last step is very laborious.

G. F. CARRIER

Asymptotic expansions. By A. Erdélyi. Dover Publications, New York, 1956. vi + 108 pp. $1.35.

This is an unaltered republication of Technical Report 3, prepared under Contract Nonr-220(11) for the Office of Naval Research.

An Essay on the foundations of geometry. B. A. W. Russell. Dover Publications, New York, 1956. xviii + 201 pp. $3.25.

Except for an added foreword by M. Kline (7 pp.), this is an unaltered edition of the 1897 edition.

Mechanik der Kontinua. By G. Hamel. Edited by I. Szabó. B. G. Teubner, Stuttgart, 1956. 210 pp. $7.10.

In spite of its sweeping title, the book is primarily concerned with mechanics of fluids. The first part (114 pp.) is devoted to perfect fluids and contains chapters on foundations, one-dimensional motions of air (sound, detonation), potential flows without free surfaces, vortex motion, steady two-dimensional flows of compressible fluids, and potential flows of compressible fluids. The second part (66 pp.) deals with viscous fluids and has chapters on the Navier-Stokes equations, slow flow, exact solutions, boundary layer theory and turbulence. The third part (19 pp.) has the heading "On more general deformable systems"; its only chapter has the title "Elastic vibrations" but is mostly devoted to the plane flow of a perfectly plastic solid. Though statements in the editor's preface would seem to contradict this impression, the last part strongly suggests that the author's death interrupted work on a more ambitious treatise, leaving us with two completed parts and miscellaneous notes.

W. Prager

Rayleigh's principle and its applications to engineering. By G. Temple and W. G. Bickley. Dover Publications, Inc., New York, 1956. 152 pp. $1.50 (paper bound).

This is an unabridged and unaltered republication of the book originally published by the Oxford University Press in 1913.

Proceedings of the conference on high-speed aeronautics. Edited by Antonio Ferri, Nicholas J. Hoff, and Paul A. Libby. Polytechnic Institute of Brooklyn, Brooklyn, New York, 1955. vi + 392 pp. $5.25.

This volume is a collection of papers by distinguished aeronautical scientists given as part of the centennial celebrations of the Polytechnic Institute of Brooklyn. Most contributions are on aerodynamical topics but important structural fields are also covered. About one half of the papers are in the nature of reviews while the remainder describe new developments.

On the aerodynamical side von Kármán gives a very broad but connected criticism of High Speed Flow Theory in its latest stage of development. Of particular interest are his presentation of linearized theory in a logical order and his lucid explanation of transonic and hypersonic flow phenomena. A second review paper by Dryden covers the complete development of the theory of transition; this emphasizes the role of spots of turbulence in causing the breakdown of laminar flow. In the group of research papers Crocco gives a theory of shock boundary layer interaction applicable to the base pressure problem, Oswatitsch and Keune describe a new method of treating transonic flow part bodies and Ferri discusses conical fields with unusual mixed flow patterns in transverse planes. There are three papers on Hypersonic Flow; two, by Libby and Bloom and by Smelt respectively, describing experimental facilities and a third, by Kantrowitz, giving an all too brief survey of physical phenomena. The remaining papers on fluid mechanics are by Busemann on the relation between noise and wave drag, by Jones on the efficient operation of high speed aircraft, by Schlicting on Cascade Theory, and by Michel and Sirieix on Transonic Lift.

On the structural side Broglio gives a complete account of the balance method in thermal stress analysis while Hoff develops a long and clear discussion of creep phenomena. Two further papers, by Duberg and Horton respectively, deal with heating effects.

M. Holt

Numerical integration of differential equations. By Albert A. Bennett, William E. Milne, and Harry Bateman. Dover Publications, Inc., New York, 1956. 108 pp. $1.35 (paper bound).

This is an unabridged and unaltered republication of a report of the Division of Physical Sciences of the National Research Council originally published in 1931.

Integral transforms in mathematical physics. By C. J. Tranter. John Wiley & Sons, Inc., New York, and Methuen & Co., Ltd., London, 1956. ix + 133 pp. $2.00.

This is a new edition of a member of the Methuen Monograph Series, first published in 1951 and reviewed in Vol. X, No. 2 of this Quarterly.

The present edition differs from the first in the addition of a chapter on dual integral equations, the correction of misprints and an enlargement of the bibliography.

The work given in the new chapter is largely of recent origin and contains many of the author's own contributions. The equations discussed arise in the solution of boundary-value problems in which the condition on one boundary is a "'mixed" one. It is welcome that an account of transform methods for such problems is thus made more widely available.

WALTER F. FREIBERGER

The theory of ordinary differential equations. By J. C. Burkill. Oliver and Boyd, Ltd., Edinburgh and London, and Interscience Publishers, Inc., New York, 1956. ix Œ 101 pp. $1.55.

This little text-book is a useful addition to the University Mathematical Texts series. The book outlines the theory of ordinary differential equations in a concise but lucid manner, with the intention of providing the basic results in the most useful rather than the most general form.

The basic existence theory for first order equations is proved in Chapter I, and Chapter II treats the general linear equation of order n. Chapter III deals with the oscillation theorems on the zeros of solutions of the homogeneous second order equation and briefly introduces eigenfunctions and expansions. The remainder of the book considers ordinary differential equations, almost exclusively linear and of the second order, in complex variables. Chapters IV, V and VI are concerned with solutions in series, singularities of equations, and solutions expressed in integrals along a contour. The main results and formulae for Legendre and Bessel functions are developed in Chapters VII and VIII. The last chapter gives a brief introduction to asymptotic series and asymptotic solutions of differential equations. A number of illustrative examples and problems with answers are given throughout the text.

Although by no means intended to replace more comprehensive books on the title subject, the text should provide a good basis for a second course on differential equations.

R. T. SHIELD

Random processes in automatic control. By J. Halcombe Laning, Jr., and Richard H. Battin. McGraw-Hill Book Co., Inc., New York, Toronto, London, 1956. ix + 434 pp. $10.00.

That this book is an outgrowth of a course given to aeronautical engineers is rather surprising. The objective is to present a background for study of automatic control of aircraft, of fire control, etc. and many of the examples lean in this direction. The main part of the test, however, describes the subject of random processes and controls in quite general terms. Approximately half of the book is devoted to probability theory and the description of random processes. Although this material is quite concentrated and would require careful reading by someone lacking preliminary training in probability or statistics, the presentation is clear, concise and well organized. It is indeed remarkable how the authors manage to give such an extensive survey of probability theory in such a short space. The mathematical rigor does not approach that of advanced mathematical texts, but the exposition certainly could not be classed as a popularized version.

The last half of the book is more applied. There is a chapter on analysis of effects of time-invariant linear systems on stationary processes (filtering and prediction with noise) and another chapter on optimum smoothing and prediction (Wiener theory) both of which are well developed subjects. Two other chapters deal with generalizations to non-stationary systems and optimum operation with finite data. Much of the material in these latter two chapters is original, but the algebra is rather involved in places. The last half of the book does not, unfortunately, measure up to the standards of the first part either as to subject matter or presentation. As a whole, however, the book is well written and a very worthwhile contribution to the literature.

GORDON F. NEWELL

Applied analysis. By Cornelius Lanczos, Prentice Hall, Inc., Englewood Cliffs, N. J., 1956. xx + 539 pp. $9.00.

Some ten years ago, satisfactory textbooks on numerical analysis in English could have been counted on the fingers of one hand. Since then, the spectacular development of high-speed computing equipment has revived the interest of mathematicians in this field. The reappearance of numerical analysis in the curricula of many colleges and universities was accompanied by the publication of several outstanding texts. These range in treatment from the strictly practical to the highly sophisticated, and in scope from collections of the authors' favorite methods to comparative surveys of methods suitable for various types of computing devices. In these terms of reference, the present book would have to be described as striking an excellent balance between the practical and the sophisticated points of view, but restricting the discussion to a rather small number of methods. This statement, however, would fail to convey an idea of the stimulating character of the book. As is indicated by their hyphenated names, many methods of numerical analysis have a long history of development by small improvements. If such a method were merely described and proved to furnish the desired result, the reader might be left wondering how the complex procedure could ever have been invented. By concentrating on principal points and letting the reader fill in the details, the present author not only gains the space needed for a more adequate coverage of the development of the various methods, but actually makes the reader feel that he has participated in this development.

Chapter I deals with algebraic equations, a "movable strip" version of Horner's scheme and Bernoulli's method of moments being at the center of the discussion. Chapter II treats matrices and eigenvalue problems with welcome emphasis on geometrical aspects. Chapter III is devoted to the large-scale linear systems that result from the algebraisation of boundary value problems. In particular, the author's "spectroscopic method" of determining the real eigenvalues of large matrices and the corresponding method of solving large systems of linear equations are developed. Chapter IV is concerned with harmonic analysis, the emphasis being on the interpolation aspects of the Fourier series. Among the special features of this chapter are discussions of the use of "σ factors" for smoothing out the Gibbs oscillations, numerical aspects of the Fourier and Laplace transforms with applications to circuit design, and an improved method of searching for hidden periodicities. Under the title "Data Analysis", Chapter V deals with interpolation and smoothing. Chapter VI treats quadrature methods, more than half of the chapter being devoted to Gaussian quadrature and several useful modifications of this. In particular, a quadrature formula is developed that employs only the values of the function and its derivatives up to a certain order at the two endpoints of the interval; the use of this formula in boundary value problems is illustrated. The final chapter "Power Expansions" emphasizes the use of Chebyshev polynomials, discussing the author's "τ-method" among other applications.

W. PRAGER

Momentum transfer in fluids. By William H. Corcoran, J. B. Opfell and B. H. Sage. Academic Press, New York, 1956. xi + 394 pp. $9.00.

This is a book mainly written for the Chemical Engineer who does not have or need much previous knowledge in the other branches of Fluid Mechanics. It consequently stresses the theoretical treatment of the phenomena encountered in internal shear flows. In Chapter I the concept of momentum transfer is introduced and then applied to a number of cases involving laminar flow. It is concluded with discussion of transition to turbulent flow and friction coefficients. Chapters II to IV deal with the properties, macroscopic characteristics and some analysis of turbulent flows, treating such topics as the similarity hypothesis, the momentum transfer hypothesis, the vorticity transfer hypothesis and their applications to flows in channels. In Chapter V the basic equations of general Fluid Mechanics are developed. Chapter VI defines and discusses such terms as correlations, spectrum, etc. in the treatment of turbulence and the final chapter introduces the reader to the concept of the boundary layer and its various applications. While the main body of the book does not make use of either vector or tensor analysis, an exhaustive and rigorous introduction to tensor analysis and the statistical theory of turbulence is added as an Appendix. It is the reviewers opinion that, while the book seems to be designed mainly for chemical engineers, some of its chapters strike such a happy compromise between mathematical treatment and physical discussion, as to warrant its use by other branches of engineering dealing with fluid mechanics.

P. MAEDER

Elements of pure and applied mathematics. By Harry Lass. McGraw-Hill Book Co., Inc., New York, Toronto, London, 1957. xi + 491 pp. $7.50.

This is another addition to the growing collection of text books that give a very brief introduction to numerous topics. The chapter headings are: 1. linear equations, determinants, and matrices, 2. vector analysis, 3. tensor analysis, 4. complex-variable theory, 5. differential equations, 6. orthogonal polynomials, Fourier series and Fourier integrals, 7. the Stieltjies integral, Laplace transform, and calculus of variations, 8. group theory and algebraic equations, 9. probability theory and statistics, 10. real-variable theory. Each of these is about 40 pages except for the last chapter that is about 100 pages.

As compared with most other books intended as text books for an advanced undergraduate or graduate level, this book covers more subjects in more condensed form and in a more abstract way although not in an extreme way. The book is partly "pure" mostly by virtue of the choice of topics; the presentation is not consistently more rigorous than other books. One annoying feature is the frequent use throughout the first nine chapters of mathematical concepts discussed in the tenth chapter. Aside from this, the book is fairly well organized and well written. It is perhaps just the type of book that some instructors would like to use, this being largely a matter of personal preference.

G. F. Newell

The theory of hydrodynamic stability. By C. C. Lin. Cambridge University Press, New York, 1955. xi + 155 pp. $4.25.

This monograph presents a unified survey of the theory of stability with respect to infinitesimal disturbances of laminar flows of a homogenous viscous fluid. This is a subject to which the author has made many fundamental contributions and to which he has been a continual source of stimulation for further work by both his students and colleagues. In addition to the presentation of the general theory and a careful exposition of the mathematical questions raised by this theory there are separate chapters on Couette, Plane Poiseuille, and Boundary Layer flows, and on the use of stability theory to problems in Astro- and Geophysics. The various contributions of the many workers in the field are specifically delineated and the book ends with an exhaustive bibliography, complete through 1955.

An important service has been performed by the careful indication throughout the book of what parts of the theory can be considered as known and what parts are controversial. This is particularly useful in view of the many physical and delicate mathematical questions that have been raised over the various simplifying assumptions and mathematical procedures that have been used in the development of the subject. The monograph should serve as an excellent source book and stimulus to anyone wishing to enter an active and interesting field. There are many sections which end with a remark such as "adequate methods.....are still lacking" or "a complete theory.....is as yet far from being completely developed."

Peter Chiarulli

General relativity and cosmology. By G. C. McVittie. John Wiley & Sons, Inc., New York, 1956. x + 198 pp. $9.00.

This is the fourth volume of the International Astrophysics Series, which aims to provide authoritative volumes, suitable for both specialists and students, dealing with the main branches of astrophysics and radio astronomy. On leafing through the book, two things catch the eye. First, a welcome rarity in technical literature, the author explains at some length his philosophical views about science. Secondly, he presents details which one is glad to have readily available. Thus, we find Dingle's formulae for the Einstein tensor calculated for any orthogonal line element, and tables of observed results for the bending of a light ray grazing the sun and for red-shifts, solar and cosmological. The introductory paragraphs to the various chapters are excellent, and it was only when I started to study the mathematics of the book that I found myself out of tune with the author.

Perhaps this is due to two related views I hold about the general theory of relativity: (1) it is a

thing of beauty, the essential simplicity of which must at all costs be made to shine through those formal manipulations which cannot be avoided in applying it, and (2) we must develop and use to the full our geometrical intuitions about space-time in the same way as mathematical physicists, through the ages, developed and used their kinematical intuitions about space and time in the Newtonian scheme. I can best describe the book in a negative sense by saying that those views are not the author's. He has no use for geometrical intuition in space-time, but, in supporting his own scepticism, he should not mislead the reader by telling him (p. 31) that in a Riemannian 2-space it is impossible to write down the curvature tensor, the Ricci tensor and the curvature invariant in terms of the Gaussian curvature; the very simple formulae which do this will be found on pp. 89, 96 of Synge and Schild, Tensor Calculus.

The following list of chapter headings will show the scope of the book: I. Introduction. II. The tensor calculus and Riemannian geometry. III. Newtonian mechanics and special relativity. IV. The principles of general relativity. V. The Schwarzschild space-time. VI. Approximations to Einstein's equation and Newtonian gas-dynamics. VII. Special cases in Newtonian gas-dynamics. VIII. Theory of uniform model universes. IX. Model universes and the system of galaxies. There are references and notes at the end, and a good index.

This is not, in my opinion, a book for a beginner in relativity. But for one who understands the theory already in its general outlines, and seeks additional special information, particularly in relation to cosmological problems, the last two chapters will provide much food for thought.

Here are some minor criticisms. The derivation of the equations of a null geodesic (p. 22) seems unsound, because it supposes falsely that all curves adjacent to a null geodesic are null to the first order. The view (p. 35) that the cosmological constant is essentially a constant of integration stems from the author's curious way of looking at the matter. In eq. (3.204) read mp for p, where m is molecular weight. It is a pity (p. 51) to cry down Minkowski's great achievement by referring to him as "one of the mathematicians who first employed" the idea of flat space-time; or is there another claimant to priority?

J. L. Synge

Introduction to statistical analysis. By Wilfrid J. Dixon and Frank J. Massey, Jr. McGraw-Hill Book Co., Inc., New York, London, Toronto, 1957. xiii + 488 pp. $6.00.

As the title might suggest, this is an undergraduate text for the non-specialist. The authors (who work in biostatistics) have been careful to select varied material from industry, educational measurements, and commercial fields, as well as from their specialty; and every problem and exercise is in language comprehensible to the average undergraduate. So many topics are touched upon, and often dismissed with so few sentences, that the "introduction" is sometimes little more than a distant glimpse. There are few demands upon mathematical proficiency. In other respects the section headings pretty well cover the range of current statistical interest, save for the notable omission of the specialized topics relating to business forecasting and index numbers. Almost a quarter of the book consists of well-chosen tables. This second edition includes a brief terminal chapter on a posteriori probability. A useful feature of the form of presentation is the separate listing of "Discussion Questions," "Class Exercises," and "Problems." A chapter on "Microstatics" deals with the theory of small samples. Many exercises carry on in succession with data found by the individual student in sampling either by use of random-number tables or from populations of tags, disks, or beads. No new principles are developed in this text.

Albert A. Bennett

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